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Quasi-Gaussian Model in QuantLib

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Agenda

» Why is it worth to look at another complex rates model?

» What are the Quasi-Gaussian model dynamics and properties?

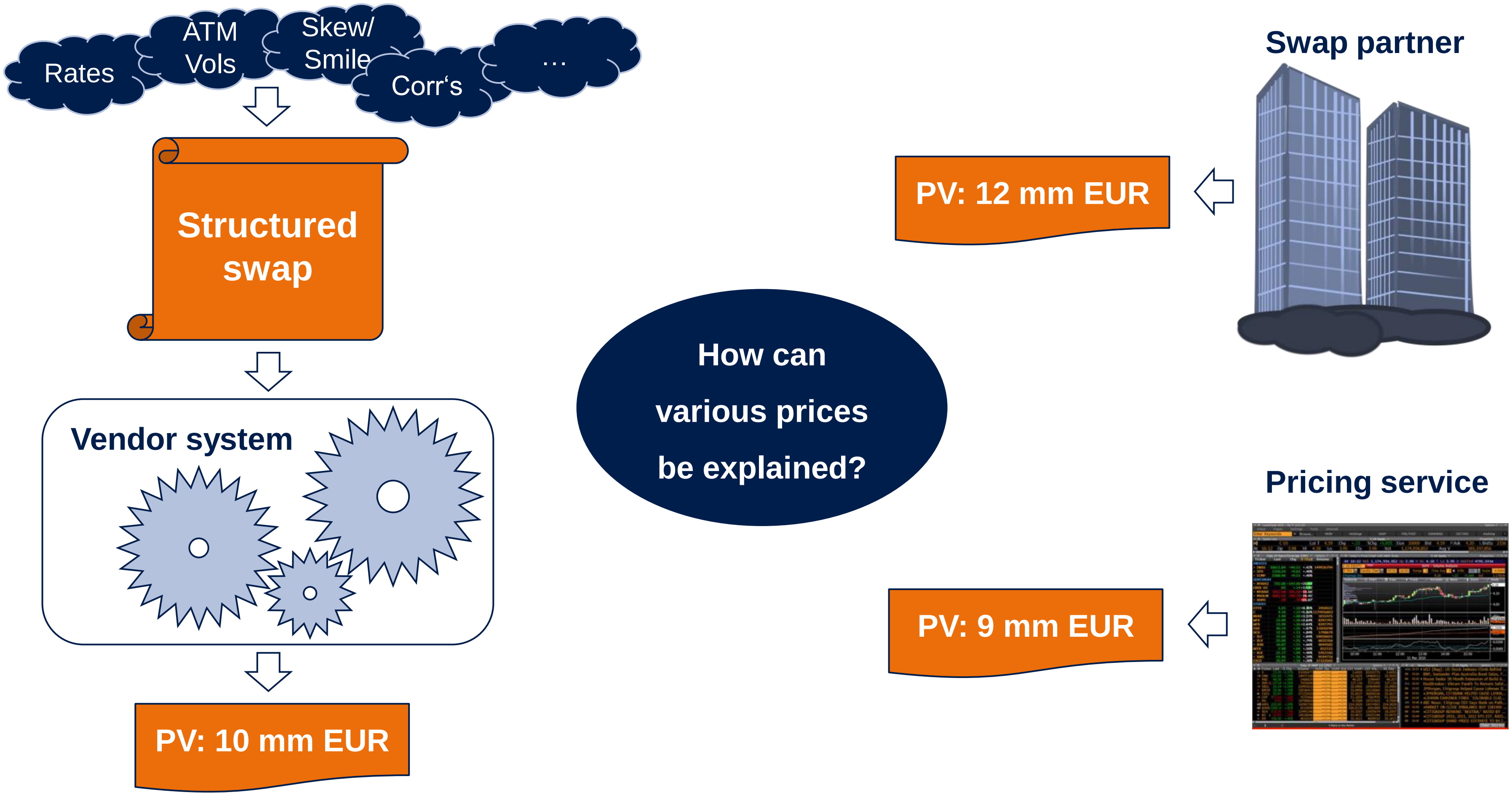
» How can the model be calibrated?

» Proof of concept by a callable CMS spread swap case study

» Summary and References

Why is it worth to look at another complex rates model?

Model validation and independent price verification exercises benefit from a flexible model class to assess various product features



Quasi-Gaussian models allow to switch on/off effects arising from the number of risk factors, volatility skew/smile and correlation

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What are the Quasi-Gaussian model dynamics and properties?

Quasi-Gaussian models may be described in terms of the scalar short rate $r(t)$, state variable $x(t)$ and auxilliary variable $y(t)$

Consider short rate $r(t)$ with dynamics[1]

$$r(t) = f(0,t) + 1^\top x(t)$$

$$dx(t) = [y(t)1 - \chi x(t)]dt + \sigma_r(t,\cdot)^\top dW(t), \qquad x(0) = 0$$

$$dy(t) = [\sigma_r(t,\cdot)^\top \sigma_r(t,\cdot) - \chi y(t) - y(t)\chi]dt, \qquad y(0) = 0$$

Model parameters

d	…	number of risk factors
$x(t) = [x_1(t), \ldots, x_d(t)]^\top$	…	state variable vector
$y(t) = \begin{bmatrix} y_{11}(t) & \ldots & y_{1d}(t) \\ \vdots & & \vdots \\ y_{d1}(t) & \ldots & y_{dd}(t) \end{bmatrix}$	…	auxilliary variable matrix
$\chi = \begin{bmatrix} \chi_1 & & \\ & \ddots & \\ & & \chi_d \end{bmatrix}$	…	diagonal matrix of mean reversion speed parameters
$\sigma_r(t,\cdot) = \begin{bmatrix} \sigma_{11}(\cdot) & \sigma_{1d}(\cdot) \\ \sigma_{d1}(\cdot) & \sigma_{dd}(\cdot) \end{bmatrix}$	…	volatility matrix – to be specified in more detail

(1) The description follows L. B. G. Andersen and V. V. Piterbarg. Interest Rate Modeling. Volume I-III. Atlantic Financial Press, 2010.

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It turns out that future yield curves and discount factors may be represented independent of the choice of volatility

Consider the auxilliary vectors of mean reversion speeds

$$h(t) = \begin{bmatrix} e^{-\chi_1 t} \\ \vdots \\ e^{-\chi_d t} \end{bmatrix}, \qquad G(t,T) = \begin{bmatrix} \left(1 - e^{-\chi_1 (T-t)}\right)/\chi_1 \\ \vdots \\ \left(1 - e^{-\chi_d (T-t)}\right)/\chi_d \end{bmatrix}$$

Then future forward rates become

$$f(t,T) = f(0,t) + h(T-t)^\top [x(t) + y(t) G(t,T)]$$

Also future zero coupon bonds (i.e. discount factors) become

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \cdot \exp\left\{ -G(t,T)^\top x(t) - \frac{1}{2} G(t,T)^\top y(t) G(t,T) \right\}$$

Future forward rates $f(t,T)$ are affine functions in terms of the risk factors $x(t)$

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Volatility matrix $\sigma_r(\cdot)$ is decomposed into stochastic volatility term $z(\cdot)$ and local volatility term $\sigma_x(\cdot)$

Volatility decomposition into stochastic and local volatility part

$$\sigma_r(t,\cdot)^\top = \sqrt{z(t)} \cdot \sigma_x(t,x,y)^\top$$

Stochastic volatility is modelled as independent CIR process

$$dz(t) = \theta \cdot [z_0 - z(t)] \cdot dt + \eta(t) \cdot \sqrt{z(t)} \cdot dZ(t), \qquad z(0) = z_0 = 1, \qquad dZ(t) \cdot dW(t) = 0$$

For local volatility modelling we choose d benchmark forward rates $f_i(t) = f(t, t+\delta_i)$ $(i = 1, \dots, d)$ and propose the following dynamics

$$df_i(t) = [\cdot] \cdot dt + \sqrt{z(t)} \cdot \lambda_i(t) \cdot [a_i(t) + b_i(t) \cdot f_i(t)] \cdot dU_i(t)$$

with $dU_i(t)$ beeing correlated with $d \times d$ correlation matrix Γ

We aim at transfering benchmark forward rate dynamics into our Quasi-Gaussian model

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Local volatility is specified based on benchmark rate volatility dynamics

Set

$$\sigma^f(t,\cdot) = \begin{bmatrix} \lambda_1(t)[a_1(t) + b_1(t)f_1(t)] & & \\ & \ddots & \\ & & \lambda_d(t)[a_d(t) + b_d(t)f_d(t)] \end{bmatrix}$$

and

$$H(t)H^f(t)^{-1} = \left[H^f(t)H(t)^{-1}\right]^{-1} = \begin{bmatrix} e^{-\chi_1\delta_1} & \dots & e^{-\chi_d\delta_1} \\ \vdots & & \vdots \\ e^{-\chi_1\delta_d} & \dots & e^{-\chi_d\delta_d} \end{bmatrix}$$

and decompose correlation matrix (e.g. by Cholesky decomposition)

$$\Gamma = D^\top D$$

Then Quasi-Gaussian local volatility becomes

$$\sigma_x(t,x,y)^\top = \left[H^f(t)H(t)^{-1}\right]^{-1} \cdot \sigma^f(t,\cdot) \cdot D^\top$$

Note that x and y enter σ_x implicitely by the future benchmark rates $f_1, \dots, f_d$ in σ^f

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We may summarize the Quasi-Gaussian dynamics which need to be implemented e.g. in a Monte Carlo simulation

$$dx(t) = [y(t)1 - \chi x(t)] \cdot dt + \sqrt{z(t)} \cdot \left[H^f(t)H(t)^{-1}\right]^{-1} \cdot \sigma^f(t,\cdot) \cdot D^\top \cdot dW(t), \qquad \mathrm{x}(0) = 0$$

$$dy(t) = \left[z(t)H(t)H^f(t)^{-1}\sigma^f(t,\cdot)\,\Gamma\,\sigma^f(t,\cdot)\left[H(t)H^f(t)^{-1}\right]^\top - \chi y(t) - y(t)\chi\right] dt, \qquad y(0) = 0$$

$$dz(t) = \theta \cdot [z_0 - z(t)] \cdot dt + \eta(t) \cdot \sqrt{z(t)} \cdot dZ(t), \qquad z(0) = z_0 = 1$$

Critical piece of a Monte Carlo simulation is the integration of the CIR process for the stochastic volatility $z(t)$

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What are properties of the various model parameters?

$$\left[H^f(t)H(t)^{-1}\right]^{-1} = \begin{bmatrix} e^{-\chi_1\delta_1} & \dots & e^{-\chi_d\delta_1} \\ \vdots & & \vdots \\ e^{-\chi_1\delta_d} & \dots & e^{-\chi_d\delta_d} \end{bmatrix}$$

- δ_i specify explicitely modelled rates; rates in between are interpolated
- χ_i specify fading speed of shocks

$$\sigma^f(t,\cdot) = \begin{bmatrix} \lambda_1(t)[a_1(t)+b_1(t)f_1(t)] & & \\ & \ddots & \\ & & \lambda_d(t)[a_d(t)+b_d(t)f_d(t)] \end{bmatrix}$$

- $\lambda_i(t)$ control overall (ATM) volatility
- $b_i(t)$ control volatility skew
- $a_i(t)$ redundant and set fixed

$$\sigma_r(t,\cdot)^\top = \sqrt{z(t)} \cdot \sigma_x(t,x,y)^\top$$

$$dz(t) = \theta \cdot [z_0 - z(t)] \cdot dt + \eta(t) \cdot \sqrt{z(t)} \cdot dZ(t)$$

- Vol-of-vol $\eta(t)$ controls volatility smile (i.e. implied vol curvature)
- θ termstructure of smile

$$\Gamma = D^\top D$$

- Correlation matrix Γ controls de-correlation of interest rates

Quasi-Gaussian model allows disentangling of the various effects which drivie interest rates

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How can the model be calibrated?

Calibration is based on deriving (approximate) swap rate dynamics in the Quasi-Gaussian model

1	Ito‘s Lemma	Use Ito‘s Lemma and write swap rate dynamics in terms of scalar Brownian motion
2	Markovian projection	Apply Markovian projection methods and derive approximate local volatility function
3	Linearization	Apply linearization (and further approximations) to derive time-dependent Heston-like dynamics
4	Parameter averaging	Use averaging techniques to derive (approximate) time-homogenuous Heston-like dynamics
5	Variable transformation	Apply variable transformation to arrive at Heston model
6	Heston model vanilla option	Finally, use semi-analytical methods to price Vanilla option in Heston model

Given a formula for Vanilla options (i.e. Swaptions) we may calibrate the Quasi-Gaussian model to observable swaption volatility market data

Step 1 - Use Ito‘s Lemma and write swap rate dynamics in terms of scalar Brownian motion

(Forward) swap rate in a multi-curve setting may be written in terms of

a) future discount factors and

b) deterministic weights capturing tenor basis spreads

$$S(t) = \frac{\sum_{i=1}^{N} L_i(t) \cdot \tau_i \cdot P(t, T_i)}{\sum_{j=1}^{M} \tau_j \cdot P(t, \bar{T}_j)} = \frac{\sum_{i=0}^{N} \omega_i \cdot P(t, T_i)}{\sum_{j=1}^{M} \tau_j \cdot P(t, \bar{T}_j)}$$

Recall that future discount factors (or zero bonds) are wttten in terms of state variable x and y

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \cdot \exp\left\{ -G(t,T)^\top x(t) - \frac{1}{2} G(t,T)^\top y(t) G(t,T) \right\}$$

Thus future swap rate is essentially a function of state variable x (and y)

$$S(t) = S(t; x, y)$$

Applying Ito‘s lemma and martingale property yields

$$dS(t) = [\cdot] \cdot dt + \nabla_x S(t) \cdot \sqrt{z(t)} \cdot \sigma_x(t,x,y)^\top \cdot dW(t)$$

$$= \sqrt{z(t)} \cdot [\nabla_x S(t) \sigma_x(t,x,y)^\top \sigma_x(t,x,y) \nabla_x S(t)^\top]^{1/2} \cdot dU^A(t)$$

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Step 2 and 3 - Apply Markovian projection methods plus linearization and derive approximate local volatility function

We approximate the general swap rate dynamics

$$dS(t) = \sqrt{z(t)} \cdot [\nabla_x S(t)\sigma_x(t,x,y)^\top \sigma_x(t,x,y)\nabla_x S(t)^\top]^{1/2} \cdot dU^A(t)$$

by expected volatility dynamics depending only on the swap rate itself

$$dS(t) \approx \sqrt{z(t)} \cdot \phi(t, S(t)) \cdot dU^A(t)$$

with ϕ defined based on conditional expectation

$$\phi(t,s)^2 = E^A\{\nabla_x S(t)\sigma_x(t,x,y)^\top \sigma_x(t,x,y)\nabla_x S(t)^\top \mid S(t) = s\}$$

Linearisation and further approximation yields

$$dS(t) \approx \sqrt{z(t)} \cdot \left[\phi\big(t, S(0)\big) + \phi_s\big(t, S(0)\big)\big(S(t) - S(0)\big)\right] \cdot dU^A(t)$$

$$\approx \sqrt{z(t)} \cdot \lambda_S(t) \cdot \left[b_S(t) \cdot S(t) + \big(1 - b_S(t)\big) \cdot S(0)\right] \cdot dU^A(t)$$

with deterministic time-dependent functions $\lambda_S(t) = \phi\big(t, S(0)\big)/S(0)$ and $b_S(t) = S(0)\phi_s\big(t, S(0)\big)/\phi\big(t, S(0)\big)$

The scalar time-dependent functions $\lambda_S(t)$, $b_S(t)$, and $z(t)$ capture all the information about the original Quasi-Gaussian model

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Step 4 - Use averaging techniques to derive (approximate) time-homogenuous Heston-like dynamics

We arrive at a two-dimensional model for the forward swap rate

$$dS(t) = \sqrt{z(t)} \cdot \lambda_S(t) \cdot \left[b_S(t) \cdot S(t) + \left(1 - b_S(t)\right) \cdot S(0)\right] \cdot dU^A(t)$$

$$dz(t) = \theta \cdot [z_0 - z(t)] \cdot dt + \eta(t) \cdot \sqrt{z(t)} \cdot dZ(t)$$

with deterministic time-dependent functions $\lambda_S(t)$ and $b_S(t)$, and $\eta(t)$

Map time-dependent parameters to time-homogenous parameters

$$\lambda_S(t) \mapsto \bar{\lambda}_S,\ b_S(t) \mapsto \bar{b}_S, \text{ and } \eta(t) \mapsto \bar{\eta}_S \text{ s.t.}$$

$$dS(t) \approx \sqrt{z(t)} \cdot \bar{\lambda}_S \cdot \left[\bar{b}_S \cdot S(t) + \left(1 - \bar{b}_S\right) \cdot S(0)\right] \cdot dU^A(t)$$

$$dz(t) \approx \theta \cdot [z_0 - z(t)] \cdot dt + \bar{\eta}_S \cdot \sqrt{z(t)} \cdot dZ(t)$$

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Step 5 and 6 - Apply variable transformation to arrive at Heston model with semi-analytical Vanilla option formula

Shift swap rate to arrive at Heston model dynamics

$$dS(t) = \sqrt{z(t)} \cdot \bar{\lambda}_S \cdot \left[\bar{b}_S \cdot S(t) + \left(1 - \bar{b}_S\right) \cdot S(0)\right] \cdot dU^A(t)$$

$$= \sqrt{z(t)} \cdot \underbrace{\bar{\lambda}_S \bar{b}_S}_{\sigma_Y} \cdot \underbrace{\left[S(t) + \frac{1 - \bar{b}_S}{\bar{b}_S} S(0)\right]}_{Y(t)} \cdot dU^A(t)$$

$$dY(t) = \sqrt{z(t)} \cdot \sigma_Y \cdot Y(t) \cdot dU^A(t)$$

$$dz(t) \approx \theta \cdot [z_0 - z(t)] \cdot dt + \bar{\eta}_S \cdot \sqrt{z(t)} \cdot dZ(t)$$

» Call/put option on $S(t)$ is equivalent to option on $Y(t)$ (with shifted strike)

» Call/put option in Heston model may be evaluated by semi-analytical methods

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How does the model capture negative rates?

Local volatility specification

$$\sigma^f(t,\cdot) = \begin{bmatrix} \lambda_1(t)[a_1(t) + b_1(t) f_1(t)] & & \\ & \ddots & \\ & & \lambda_d(t)[a_d(t) + b_d(t) f_d(t)] \end{bmatrix}$$

allows modelling negative rates down to

$$f_i(t) > -a_i(t)/b_i(t)$$

However, swap rate dynamics for calibration based on convex combination of $S(t)$ and $S(0)$

$$dS(t) = \sqrt{z(t)} \cdot \lambda_S(t) \cdot \left[b_S(t) \cdot S(t) + \left(1 - b_S(t)\right) \cdot S(0)\right] \cdot dU^A(t)$$

with $\lambda_S(t) = \phi\big(t, S(0)\big)/S(0)$ and $b_S(t) = S(0)\phi_s\big(t, S(0)\big)/\phi\big(t, S(0)\big)$

Require $S(0) > 0$.

Remediation Ideas (still work in progress)

» Adapt averaging techniques directly to local vol structure $\left[\phi\big(t, S(0)\big) + \phi_s\big(t, S(0)\big)\big(S(t) - S(0)\big)\right]$

» Shift forward curve and implied normal vols (*volatility transformation*) and then apply calibration

To circumvene difficulties with negative rates for the moment we shift all yield curves by +3% in forthcoming examples

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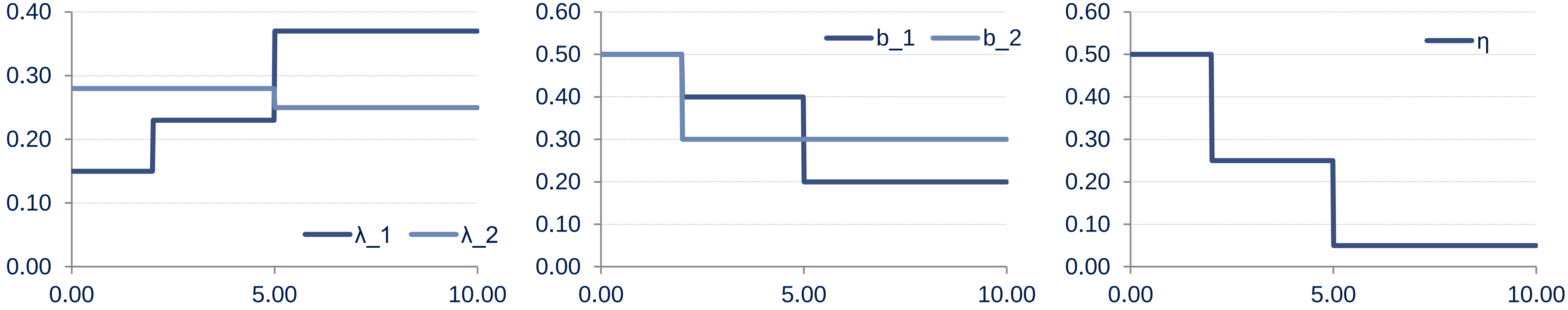
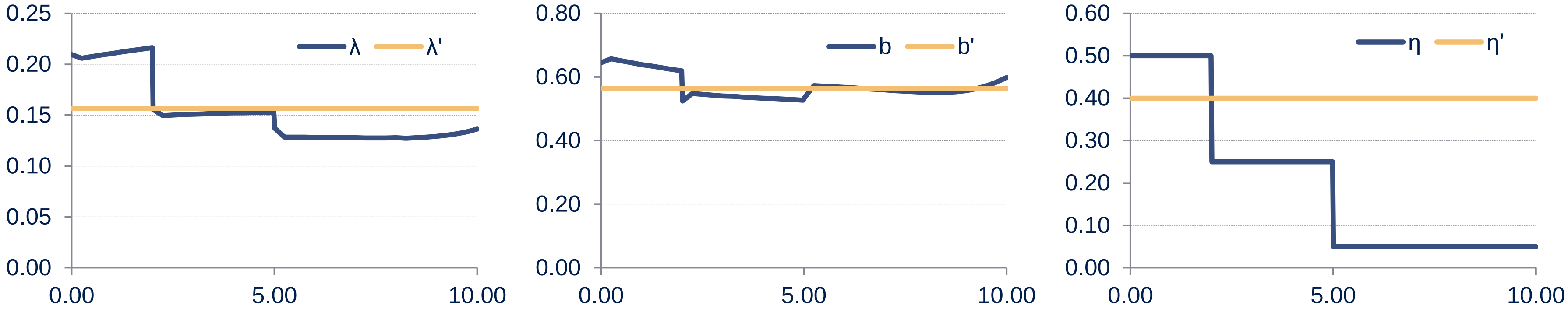
We mark a 2-factor Quasi-Gaussian Model to fit observed market volatilities

Input volatility parameters for 2-factor Quasi-Gaussian model



Derived approximate 10y x 10y swap rate volatility parameters



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How is the fit to market smiles?(1)

(1) Manual fit via analytic formula to market smile and shifted curves

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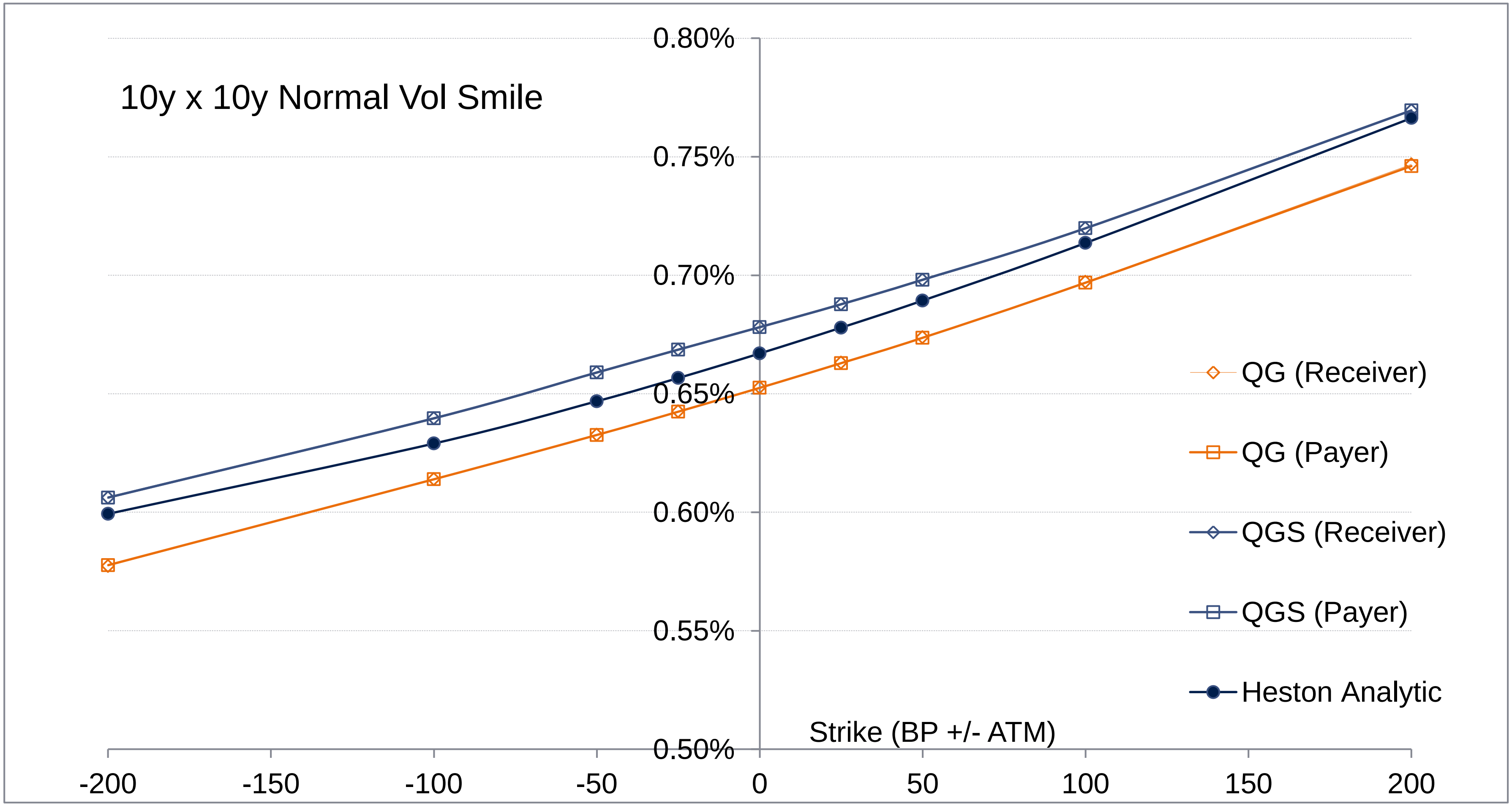
How accurate are all these approximations?



There are manageable variances between Quasi-Gaussian model, approximate Swaption model and Heston-like model

Proof of concept by a callable CMS spread swap case study

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We consider pricing of a callable CMS spread swap and analyse the impact of the various model parameters[1]

Legs	Receive	Pay
Notional	10.000 EUR	
Effective Date	2d	
Termination Date	10y	
Tenor	3m	
Payoff	Max{ 3 x [CMS10y – CMS2y], 0 }	3m Euribor – 100bp
Conventions	mod. following, Act/360	
Call Schedule	1y to 9y, annually	

Modelling scenarios

1-F Gaussian model	2-F Gaussian model w/ perfect correlation	2-F Gaussian model w/ 50% correlation	2-F QG model w/ skew	2-F QG model w/ skew & smile
» General impact of stochastic rates	» Capturing short-term and long-term shocks » ATM vol calibration	» Decoupling short-term and long-term shocks	» Capturing implied volatility skew » Improve vol calibration	» Capturing implied volatility smile (curvature » Improve calibration

It is fairly reasonable that (de-)correlation is of particular importance. But what about skew and smile?

(1) We use market data as of July 2016 but shift curves by +3% to circumvene difficulties with negative rates for our example

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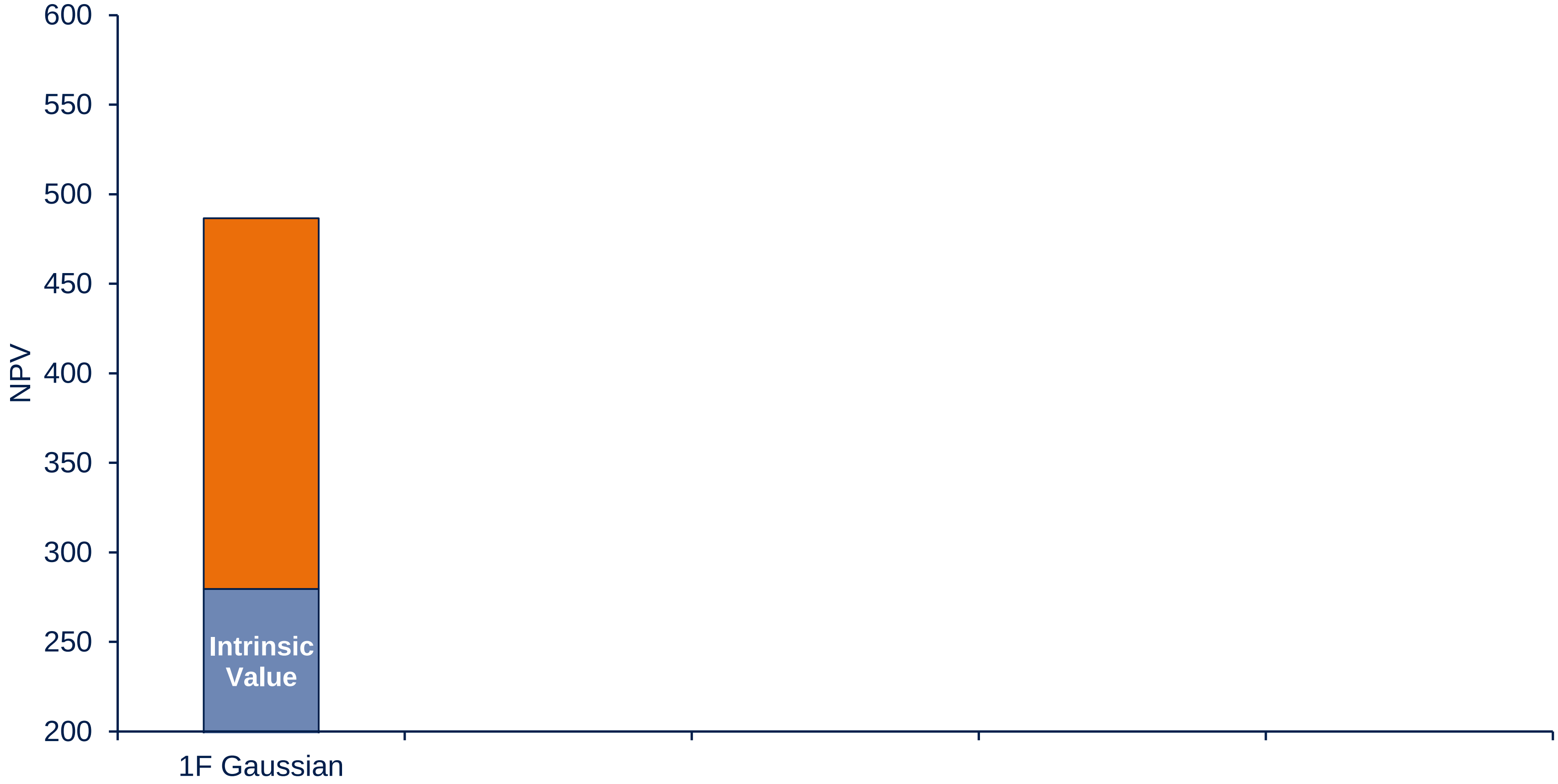
1-Factor Gaussian model in general may not capture ATM vols for both 2y and 10y swap rates

1-F Gaussian model allows differentiating general stochastic rates impact from derivative‘s intrinsic value



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2-F Gaussian model w/ perfect correlation allows improved fit to ATM volatilities

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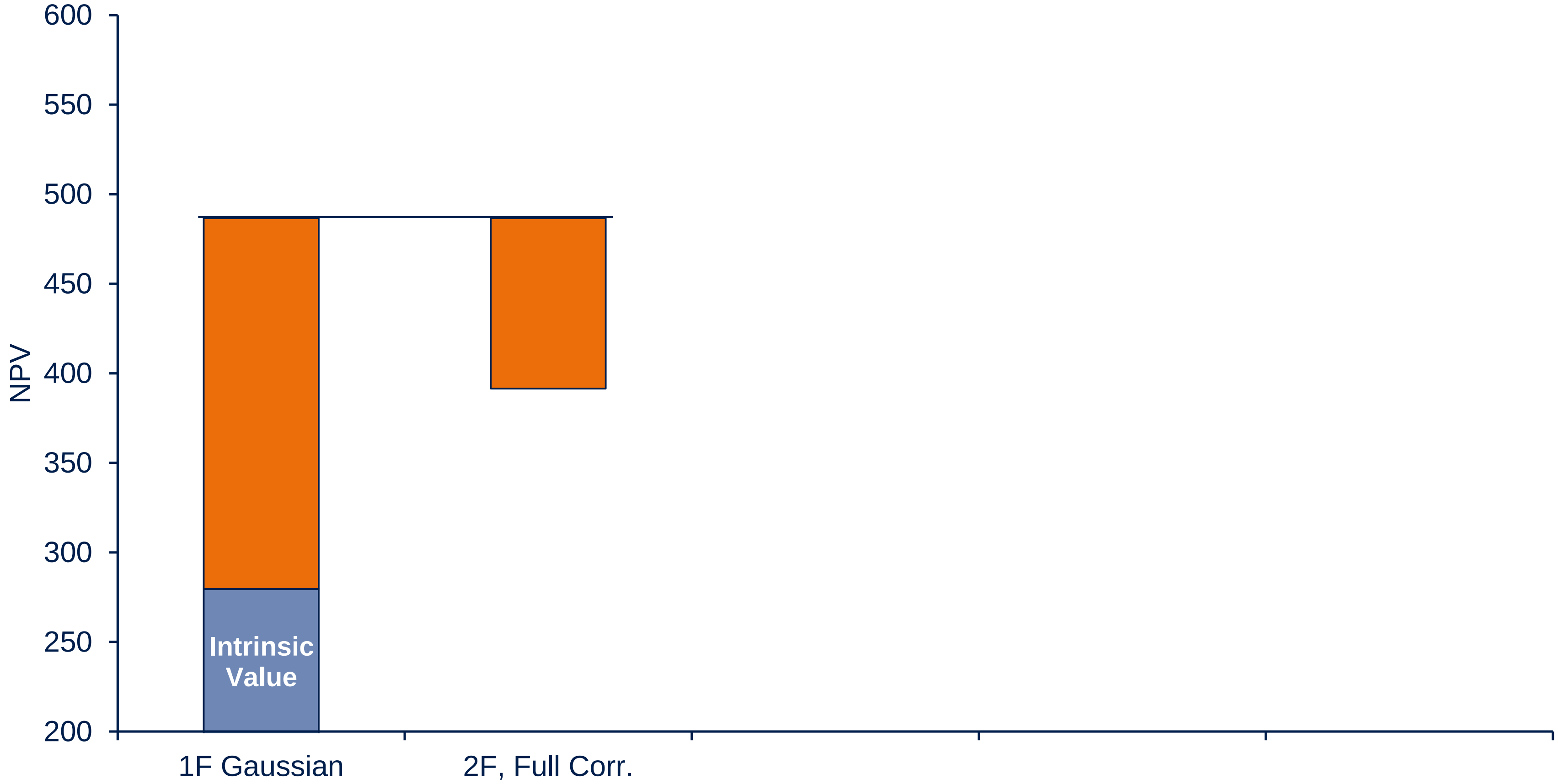
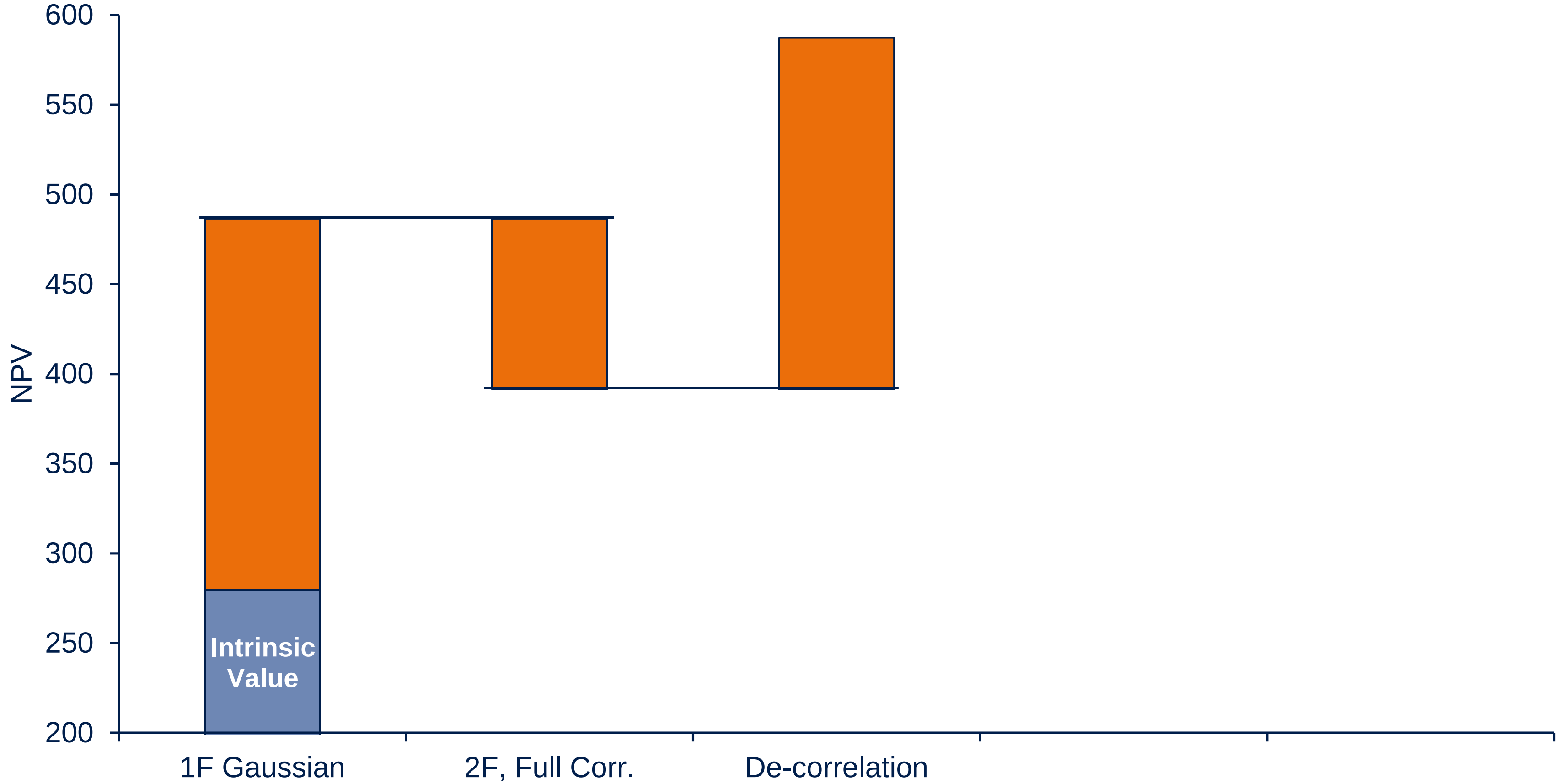
For 2-F Gaussian model w/ perfect correlation the reduction in callable note NPV is mainly driven by reduced option value



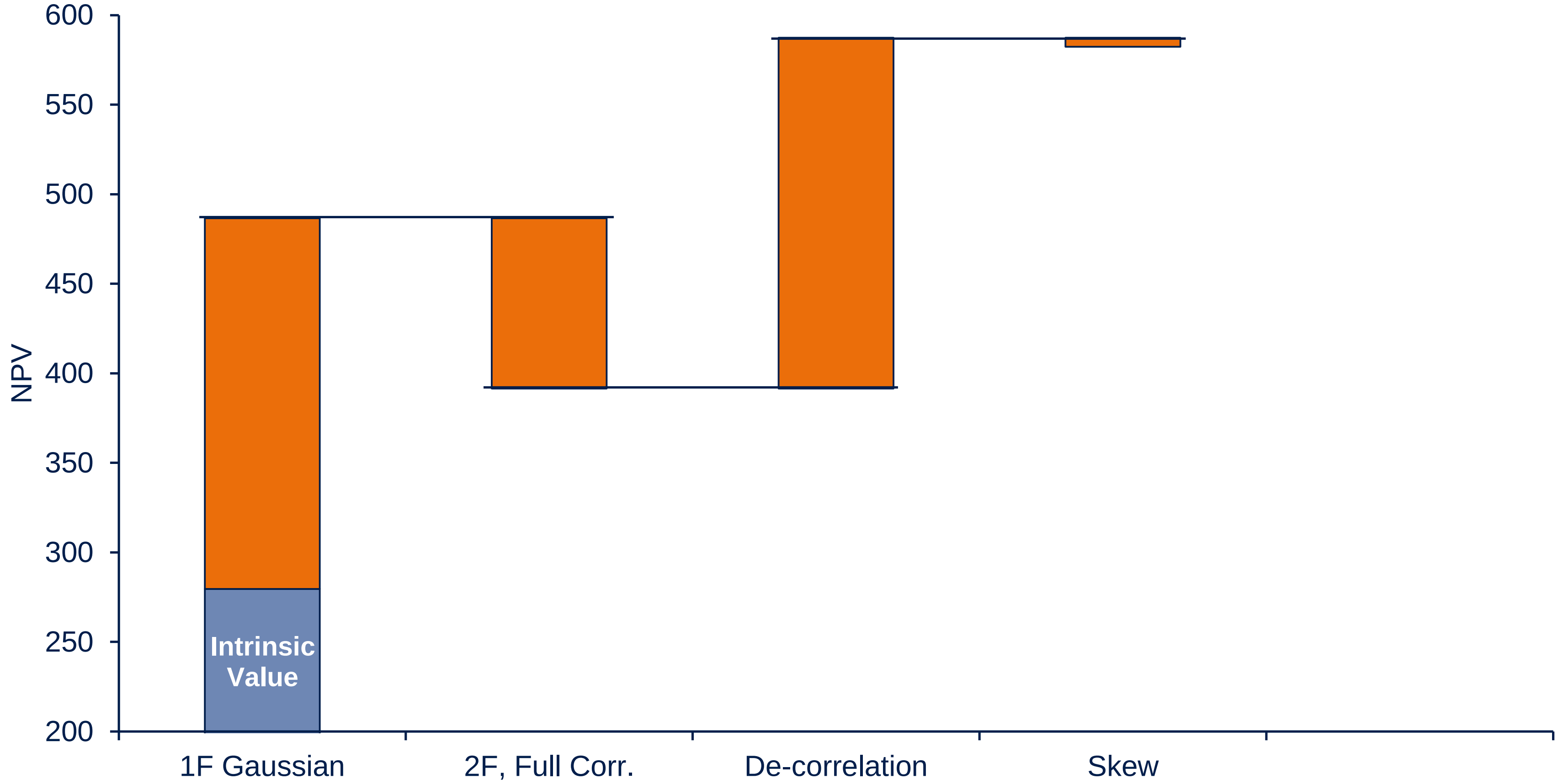
2-F Gaussian model w/ 50% model correlation yields 62% model-implied correlation between 2y vs. 10y swap rates

De-correlation in 2-F Gaussian model boosts CMS spread leg NPV



Incorporating local volatility allows capturing volatility skew

Reduced low-strike volatility reduces CMS spread leg; however effect is mainly offset by call option (i.e. option on opposite deal)



Incorporating stochastic volatility allows capturing volatility smile (i.e. curvature in implied vols)

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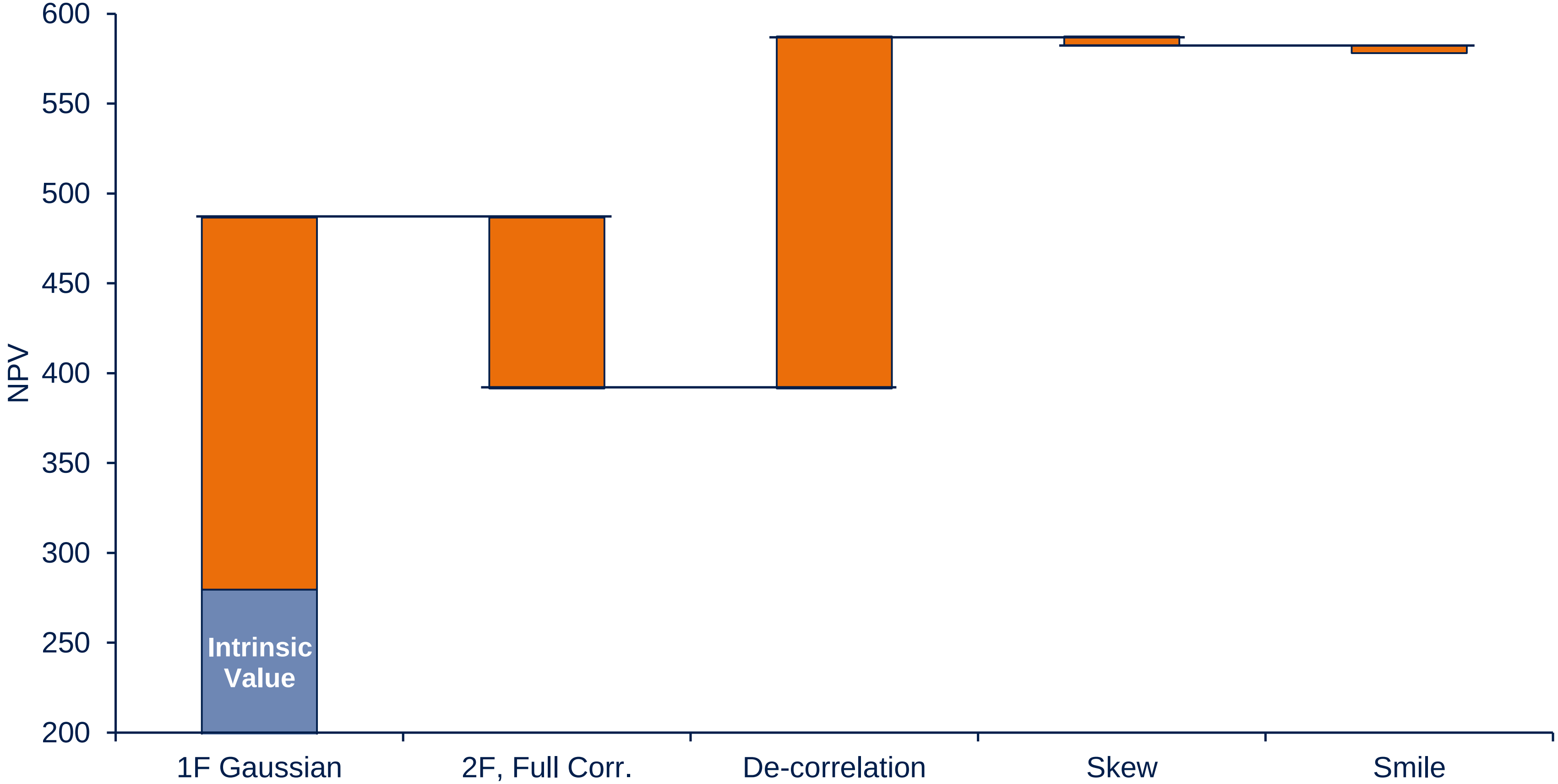
Low-strike vols are increased by stochastic volatility; again with offsetting effects on CMS spread leg and call option



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The component prices help for a detailed analysis of pricing results

Scenario	Intrinsic	1F Gaussian	2F, Full Corr.	De-Corr.	Skew	Smile
MC Pricing						
CMS2y	3,055	3,077	3,074	3,074	3,075	3,076
CMS10y	3,519	3,602	3,607	3,604	3,603	3,609
Euribor	2,753	2,754	2,756	2,751	2,757	2,757
StructLeg	1,394	1,574	1,600	1,760	1,744	1,756
FundLeg	-1,872	-1,873	-1,875	-1,870	-1,876	-1,876
Underlying	-479	-299	-275	-110	-132	-119
AMC Pricing						
NoteNPV	280	487	391	587	582	578
UnderlyingNPV	-479	-298	-275	-110	-133	-121
OptionNPV	758	784	667	697	715	699

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Summary and References

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Summary and References

Summary

- Quasi-Gaussian model appears to be a powerfull tool for model validation of comlex rates derivatives
- All relevant methods are exported to Excel with sample spread sheets available
- Further analysis/research required (negative rates, automated calibration) to get it fully functional in a production setting

References

- L. Andersen and V. Piterbarg. *Interest rate modelling, volume I to III*. Atlantic Financial Press, 2010.
- https://github.com/sschlenkrich/QuantLib/tree/master/ql/experimental/templatemodels

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