Inflation Modeling

Quaternion Risk Management

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- Region (UK, US, AU, FR, EU)
- InflationIndex
  ZeroInflationIndex, YoYInflationIndex
- InflationTermStructure
  ZeroInflationTermStructure,
  YoYInflationTermStructure,
  InterpolatedZeroInflationCurve,
  InterpolatedYoYInflationCurve,
  PiecewiseZeroInflationCurve (ZeroInflationSwapHelper),
  PiecewiseYoYInflationCurve (YoYInflationSwapHelper)
- Inflation volatility surfaces
  CPIVolatilityTermStructure,
  YoYOptionletVolatilityTermStructure (check experimental for more!)
- Instruments
  ZeroCouponInflationSwap
  YoYInflationSwap
  CPICapFloor
  YoYInflationCapFloor
- Pricing Engines
  YoYInflationCapFloorEngine (Bachelier, Black, Unit-Displaced Black)
  CPICapFloorEngine (expr: intrep. of the CPICapFloorTermPriceSurface)

Q : What would one need more?

An inflation model
- to price structured inflation products
- to simulate inflation as a risk factor (CVA, DVA etc...)

$\rightarrow$ Jarrow-Yildirim Model

Foreign currency analogy:

- nominal rate: HW1F
- real rate: HW1F
- CPI: GBM

under Q:

$$
\begin{aligned}
dn &= \lambda_n \left[\varphi_n(t) - n(t)\right] \, dt + \sigma_n \, dW_n^Q \\
dr &= \lambda_r \, \left[\varphi_r(t) - r(t)\right] \, dt + \sigma_r \, dW_r^Q - \rho_{rc} \, \sigma_c \, \sigma_r \, dt \\
dc/c &= \left[n(t) - r(t)\right] \, dt + \sigma_c \, dW_c^Q \\
dW_a^Q \, dW_b^Q &= \rho_{ab} \, dt, \qquad a, b \in \{n, r, c\}
\end{aligned}
$$

LGM

One factor LGM in a nutshell

Given the Hull-White dynamics of the nominal rate process $n(t)$ under Q

$$\begin{aligned} n(t) &= x(t) + \mu(t) \\ d\mu_n(t) &= (\theta_n(t) - \lambda_n(t)\mu(t))\,dt \\ dx(t) &= -\lambda_n(t)x(t)\,dt + \sigma_n(t)\,dW_n(t), \quad x(0) = 0. \end{aligned}$$

One can drive the LGM dynamics

$$dz(t) = \alpha(t)\,dW_n^N(t)$$

where W^z is a Brownian motion under the LGM measure, equipped with the LGM numeraire

$$N(t) = \frac{1}{P_n(0,t)} \exp\left\{H_n(t)\,z(t) + \frac{1}{2}H_n^2(t)\,\zeta(t)\right\}$$

where $\alpha_n(t) > 0$ and $H_n(t) > 0$

$$H_n(t) = \int_0^t e^{-\int_0^u \lambda_n(s)ds}\,du, \quad \alpha_n(t) = \sigma_n(t)\,e^{\int_0^t \lambda_n(u)du}, \quad \zeta_n(t) = \int_0^t \alpha_n^2(u)du$$

LGM
Pricing Recipe

Pricing formula

$$\frac{V(t)}{N(t)} = \mathbb{E}_N \left[\frac{V(T)}{N(T)} \middle| z_t \right]$$

Zero Bond value is given by

$$\begin{aligned} P(t,T,z(t)) &= \mathbb{E}_N \left[\frac{n(t)}{N(T)} \middle| z_t \right] \\ &= \frac{P(0,T)}{P(0,t)} \exp\left\{ -(H(T)-H(t))\, z(t) - \frac{1}{2}(H^2(T)-H^2(t))\,\zeta(t) \right\} \end{aligned}$$

Reduced zero bond value

$$\widetilde{P}(t,T,z(t)) = \frac{P(t,T,z(t))}{N(t)} = P(0,T)\ \exp\left\{ -H(T)\, z(t) - \frac{1}{2}H^2(T)\,\zeta(t) \right\}$$

Forward rate and short rate directly follows

$$\begin{aligned} f(t,T) &= f(0,T) + z(t)\, H'(T) + \zeta(t)\, H'(T)H(T) \\ r(t) = f(t,t) &= f(0,t) + z(t)\, H'(t) + \zeta(t)\, H'(t)H(t) \end{aligned}$$

JY Model under LGM

Model

JY Dynamics under (nominal) LGM measure:

$$
\begin{aligned}
dz &= \alpha_z \, dW_z^N, \\
dy &= \theta(t)dt + \alpha_y(t) \, dW_y^N, \\
dc/c &= \mu_c(t) \, dt + \sigma_c(t) \, dW_c^N,
\end{aligned}
$$

where $d\langle W_z \, , W_y \rangle = \rho_{zy} \, dt, \quad d\langle W_z \, , W_c \rangle = \rho_{zc} \, dt, \quad d\langle W_y \, , W_c \rangle = \rho_{yc} \, dt.$

Please note that $\theta(t)$ and $\mu_c(t)$ are determined by no arbitrage conditions and expectations and variances of state variable are available in closed form.

nominal zero coupon bond

$$\begin{aligned}P_n(t,T) =& E_N\left(\frac{N(t)}{N(T)}\Big|\mathcal{F}_t\right)\\ =& \frac{P_n(0,T)}{P_n(0,t)}\ \exp\left\{-(H_z(T)-H_z(t))\,z(t)-\frac{1}{2}(H_z^2(T)-H_z^2(t))\,\zeta(t)\right\}\end{aligned}$$

real zero coupon bond

$$\begin{aligned}P_r(t,T) =& E_N\left(\frac{I(T)}{I(t)}\frac{N(t)}{N(T)}\Big|\mathcal{F}_t\right)\\ =& \frac{P_r(0,T)}{P_r(0,t)}\exp\left\{-\,(H_y(T)-H_y(t))\,y(t)-\frac{1}{2}\left(H_y^2(T)-H_y^2(t)\right)\zeta_y(t)\right\}\end{aligned}$$

JY Model under LGM

Zero Coupon Inflation Indexed Cap Floor

class CPICapFloorEngineLGMJY : public CPICapFloor::engine

$$
\begin{aligned}
ZCIICF(t, T_0, T, K, w) =& \mathbb{E}_N \left(\frac{N(t)}{N(T)} \left[w \left(\frac{I(T)}{I(T_0)} - (1+k)^{T-T_0} \right) \right]^+ \right) \\
=& \frac{N(t)}{I(T_0)} \mathbb{E}_N \left(\frac{1}{N(T)} \left[w \left(I(T) - \widetilde{K} \right) \right]^+ \Bigg| \mathcal{F}_t \right)
\end{aligned}
$$

where $K = I(T_0)\,(1+k)^{T-T_0}$.

$$
\begin{aligned}
ZCIICF(t, T_0, T, K, w) =& w \frac{P(t,T)}{I(T_0)} \left[p\, e^{M_A + 0.5\, V_A + CV_{AB}}\, \Phi \left(w \frac{M_A + \log(p) - \log(K) + CV_{AB} + V_A}{\sqrt{V_A}} \right) \right. \\
& \left. - K\, \Phi \left(w \frac{M_A + \log(p) - \log(K) + CV_{AB}}{\sqrt{V_A}} \right) \right]
\end{aligned}
$$

where M_A, V_A, are mean and variance of A and CV_{AB} is covariance of A and B.

$$
\begin{aligned}
A &:= \int_t^T \left(n(u) - r(u) - 0.5\,\sigma_c^2 + H_z(u)\,\alpha_z(u)\,\sigma_c\,\rho_{zc} \right) du + \int_t^T \sigma_c dW_x(u) \\
B &:= -H_z(T) z(T)
\end{aligned}
$$

JY Model under LGM

Year on Year Inflation Indexed Cap Floor

class YoYCapFloorEngineLGMJY: public YoYCapFloor::engine

$$
\begin{aligned}
YoYIICF(t, T_0, T, K, w) =& \sum_{i=1}^{N} \mathbb{E}_N \left(\frac{N(t)}{N(T_i)} \left[w \left(\frac{I(T_i)}{I(T_{i-1})} - 1 - k \right) \right]^+ \right) \\
=& N(t) \sum_{i=1}^{N} \mathbb{E}_N \left(\frac{1}{N(T_i)} \left[w \left(\frac{I(T_i)}{I(T_{i-1})} - K \right) \right]^+ \middle| \mathcal{F}_t \right)
\end{aligned}
$$

where $K = 1 + k$

$$
\begin{aligned}
YoYIICF(t, T_0, T, K, w) =& w \sum_{i=1}^{N} P(t, T_i) \Bigg[e^{M_{A_i} + 0.5\, V_{A_i} + CV_{A_i B_i}} \Phi \left(w \frac{M_{A_i} + \log(p) - \log(K) + CV_{A_i B_i} + V_{A_i}}{\sqrt{V_{A_i}}} \right) \\
& - K\, \Phi \left(w \frac{M_{A_i} + \log(p) - \log(K) + CV_{A_i B_i}}{\sqrt{V_{A_i}}} \right) \Bigg]
\end{aligned}
$$

where M_{A_i}, V_{A_i}, M_{B_i}, $CV_{A_i B_i}$ can be calculated similarly as in ZCIIFC case.

We have a multi currency inflation framework to simulate the risk factor evolutions of

- nominal rate process, linked to z_i, for $i = 0, \dots n$,
- FX rate processes x_i for $i = 1, \dots n$,
- real rate processes, linked to y_i, for $i = 0, \dots m$,
- CPI processes c_i, for $i = 0, \dots m$

whose dynamics are given under the domestic LGM measure:

$$
\begin{aligned}
dz_i &= \epsilon_i \gamma_i^z \, dt + \alpha_i^z \, dW_i^z, \qquad i = 0, \dots, n, \ \ \epsilon_i = 1, \text{for } i \neq 0 \\
dx_i / x_i &= \mu_i^x(t) \, dt + \sigma_i^x \, dW_i^x, \qquad i = 1, \dots, n \\
dy_i &= \epsilon_i \beta_i^y(t) \, dt + \theta_i^y dt + \alpha_i^y(t) \, dW_i^y, \qquad i = 0, \dots, m, \ \ \epsilon_i = 1, \text{for } i \neq 0 \\
dc_i / c_i &= \mu_i^c(t) \, dt + \sigma_i^c(t) \, dW_i^c, \qquad i = 0, \dots, m
\end{aligned}
$$

where

$$
\begin{aligned}
& d\langle W_i^z, W_i^z \rangle = \rho_{ij}^{zz} \, dt, \quad d\langle W_k^x, W_l^x \rangle = \rho_{kl}^{xx} \, dt, \quad d\langle W_i^z, W_k^x \rangle = \rho_{ik}^{zx} \, dt \\
& d\langle W_i^y, W_j^y \rangle = \rho_{ij}^{yy} \, dt, \quad d\langle W_k^c, W_l^c \rangle = \rho_{kl}^{cc} \, dt, \quad d\langle W_i^y, W_k^c \rangle = \rho_{ik}^{yc} \, dt \\
& d\langle W_i^z, W_j^y \rangle = \rho_{ij}^{zy} \, dt, \quad d\langle W_k^z, W_l^c \rangle = \rho_{kl}^{zc} \, dt, \quad d\langle W_i^y, W_k^x \rangle = \rho_{ik}^{yx} \, dt
\end{aligned}
$$

Please note that the drift terms $\gamma_i^z(t)$, $\mu_i^x(t)$, $\beta_i^y(t)$, $\theta_i^y(t)$ and $\mu_i^c(t)$, are determined by no arbitrage conditions.

Implementation in QuantExt

- InflationTermStructure
  class RealRateStructure : public ZeroYieldStructure
- Models
  - class LGMJY : public CalibratedModel
  - class MultiCcyInflationLGMJYModel : public Observer, public virtual Observable
- Calibration helpers
  - class CPICapFloorHelper : public CalibrationHelper
  - class YoYCapFloorHelper : public CalibrationHelper
- Path Generators:
  -class LGMJYPathGenerator
  -class MultiCcyInflationLGMJYPathGenerator
  -Moreover: Single step, SDE simulations
- Pricing engines
  - class CPICapFloorEngineLGMJY : public CPICapFloor::engine
  - class YoYCapFloorEngineLGMJY : public YoYInflationCapFloor::engine
  - class CPICapFloorEngine : public CPICapFloor::engine (Bachelier, Black, Unit-Displaced Black)
- Optimiser
  -class ASA : public OptimizationMethod

Optimiser
ASA

Local optimizers: "From the starting point go immediately down-hill, as far as you can go."

We have implemented a global optimiser; Adaptive Similulated Annealing (ASA) in a QuantLib fashion, i.e. EndCriteria and Constraints can be invoked.

In ASA: You can go up-hill as well as down hill.

It makes random search in the parameter range and assigns probabilities for changing from one state to another.

Changing from a "better state" to a "worse state" is possible in favor of finding a global minimum.

We assume that the nominal rate process is already calibrated to cap/swaptions and concentrate only on the calibration of the inflation related processes

- Estimate historically the correlations $\rho_{nc}, \rho_{nr}, \rho_{rc}$.
- Estimate historically volatility of real rate process α_y
- Iterative calibration to inflation cap markets:
  1. Calibrate **piecewise linear** H_y's to CPI CapFloors
  2. Calibrate **piecewise flat** σ_c's to YoY CapFloors

Please note that the initial inflation term structure is matched by the construction of the model.

Calibration

Calibration Results


```
Calibrate LGMJY model
ASA dimension is 8
0. iteration calibrate H_r
CPI vol 0 2Y  1.870794 %
CPI vol 1 5Y  3.612773 %
CPI vol 2 10Y  5.549349 %
CPI vol 3 15Y  6.172859 %
CPI vol 4 20Y  6.902481 %
CPI vol 5 30Y  7.791427 %
CPI vol 6 40Y  8.476374 %
CPI vol 7 50Y  9.161320 %
ASA::minimize called
0. iteration calibration of H_r  is done
1 : model npv 0.010957: market npv 0.010957, model implied vol 1.87079 %, market vol 1.87079 % (+0.00000 %)
2 : model npv 0.035091: market npv 0.035091, model implied vol 3.61277 %, market vol 3.61277 % (-0.00000 %)
3 : model npv 0.077872: market npv 0.077872, model implied vol 5.54935 %, market vol 5.54935 % (-0.00000 %)
4 : model npv 0.10632: market npv 0.10632, model implied vol 6.17284 %, market vol 6.17286 % (-0.00002 %)
5 : model npv 0.13854: market npv 0.13854, model implied vol 6.90248 %, market vol 6.90248 % (-0.00000 %)
6 : model npv 0.18763: market npv 0.18763, model implied vol 7.79143 %, market vol 7.79143 % (+0.00000 %)
7 : model npv 0.24363: market npv 0.24363, model implied vol 8.47637 %, market vol 8.47637 % (-0.00000 %)
8 : model npv 0.30606: market npv 0.30606, model implied vol 9.16132 %, market vol 9.16132 % (-0.00000 %)
calibration error: 0.0000 %
LGM-JY A: [ 0.00581574; 0.00654461; 0.00726453; 0.00856156; 0.00973334; 0.0102054; 0.00989779; 0.00853894; 0.00886059; 0.00886059 ] [ 0.006 ]
LGM-JY H: [ 0.03 ] [ 6.58862; 14.7089; 24.6837; 30.165; 35.1278; 41.8419; 47.0785; 52.2714 ]
LGM-JY SigmaCPI: [ 0.01 ]
LGM-JY Correlations: [ 0.72 ] [ 0.17 ] [ 0.24 ]
0. iteration calibrate SigmaCPI
YoY vol 0 10Y  3.493900 %
YoY vol 1 15Y  3.133800 %
YoY vol 2 20Y  3.128700 %
YoY vol 3 30Y  3.152500 %
YoY vol 4 40Y  3.163100 %
YoY vol 5 50Y  3.173700 %
ASA::minimize called
0. iteration: calibration of SigmaCPI  is done
1 : model npv 0.0011671: market npv 0.0011671, model implied vol 3.49391 %, market vol 3.49390 % (+0.00001 %)
2 : model npv 0.0011428: market npv 0.0011428, model implied vol 3.13378 %, market vol 3.13380 % (-0.00002 %)
3 : model npv 0.001162: market npv 0.001162, model implied vol 3.12870 %, market vol 3.12870 % (+0.00000 %)
4 : model npv 0.001019: market npv 0.001019, model implied vol 3.15250 %, market vol 3.15250 % (+0.00000 %)
5 : model npv 0.00083498: market npv 0.00083498, model implied vol 3.16310 %, market vol 3.16310 % (-0.00000 %)
6 : model npv 0.0007208: market npv 0.0007208, model implied vol 3.17370 %, market vol 3.17370 % (+0.00000 %)
calibration error: 0.0000 %
LGM-JY A: [ 0.00581574; 0.00654461; 0.00726453; 0.00856156; 0.00973334; 0.0102054; 0.00989779; 0.00853894; 0.00886059; 0.00886059 ] [ 0.006 ]
LGM-JY H: [ 0.03 ] [ 6.58862; 14.7089; 24.6837; 30.165; 35.1278; 41.8419; 47.0785; 52.2714 ]
LGM-JY SigmaCPI: [ 0.0135229; 0.0145021; 0.0202875; 0.027046; 0.03437; 0.0490694 ]
LGM-JY Correlations: [ 0.72 ] [ 0.17 ] [ 0.24 ]
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Thank you for your attention!