(J)CIR(++) Hazard Rate Model

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Background

Client wants benchmarking and validation of CVA models.

Our goal is to **validate** the existing models.

Furthermore **benchmark** against alternative models.

Performance measure are historical backtests.

We differentiate between asset classes, in this case credit.

Model choice

We are aiming to model the stochastic intensity having direct control on spread dynamics (as opposed to structural models).

We like to have the following model features:

- mean reverting feature
- non negativity
- analytical tractability (survival probability)
- analytical tractability (underlying distribution)

JCIR++ Hazard Rate Model

CIR++ enhanced by a jump process:

$$\begin{aligned} \lambda(t) &= y(t) + \psi(t) \\ dy(t) &= a\left(\theta - y(t)\right) dt + \sigma_\lambda \sqrt{y(t)}\, dW(t) + dJ_{\alpha,\gamma}(t) \end{aligned}$$

$J(t)$ is a compound Poisson process

$$J(t) = \sum_{i=1}^{N(t)} S_i$$

where the number of jumps n in any time interval $(t, t+\tau)$ follows a Poisson distribution with intensity α

$$PDF(n) = \frac{e^{-\alpha\tau}\,(\alpha\tau)^n}{n!},$$

and the jump sizes s have exponential distribution with mean γ,

$$PDF(s) = \frac{1}{\gamma}\, e^{-s/\gamma}$$

JCIR: Zero Bond aka Survival probability

JCIR Zero Bond or Survival probability

$$
\begin{aligned}
P^{JCIR}(t,T;y) &= \mathbb{E}\left[e^{-\int_t^T y(s)\,ds}\right] \\
&= \mathcal{A}^{JCIR}(t,T)\, e^{-\mathcal{B}(t,T)\,y(t)} \\
\mathcal{A}^{JCIR}(t,T) &= \mathcal{A}^{CIR}(t,T) \left[\frac{2h\, e^{(a+h+2\gamma)(T-t)/2}}{2h+(a+h+2\gamma)(e^{(T-t)h}-1)}\right]^{\frac{2\alpha\gamma}{\sigma^2-2a\gamma-2\gamma^2}} \\
\mathcal{B}(t,T) &= \frac{2(e^{(T-t)h}-1)}{2h+(a+h)(e^{(T-t)h}-1)} \\
h &= \sqrt{a^2+2\sigma^2}
\end{aligned}
$$

Model class within QuantLib

The original QuantLib model classes **CoxIngersollRoss** and **ExtendedCoxIngersollRoss** are used as an inspiration.

We built our own class JCIR, derived from **CalibratedModel**, in order to use the calibration functionality.

We dropped **OneFactorAffineModel** inheritance to avoid confusion between discount bond and survival probability. There is also Tree and ShortRateDynamics functionality, which we don't need in this form.

Model class methods

- Survival Probability and its components
- CDS option pricing components
- Characteristic function and densities
- Model parameter
- Model implied Default Probability Termstructure
- Feller condition / non negativity constraint: $2\,a\,\theta > \sigma^2$

Pricing Engine

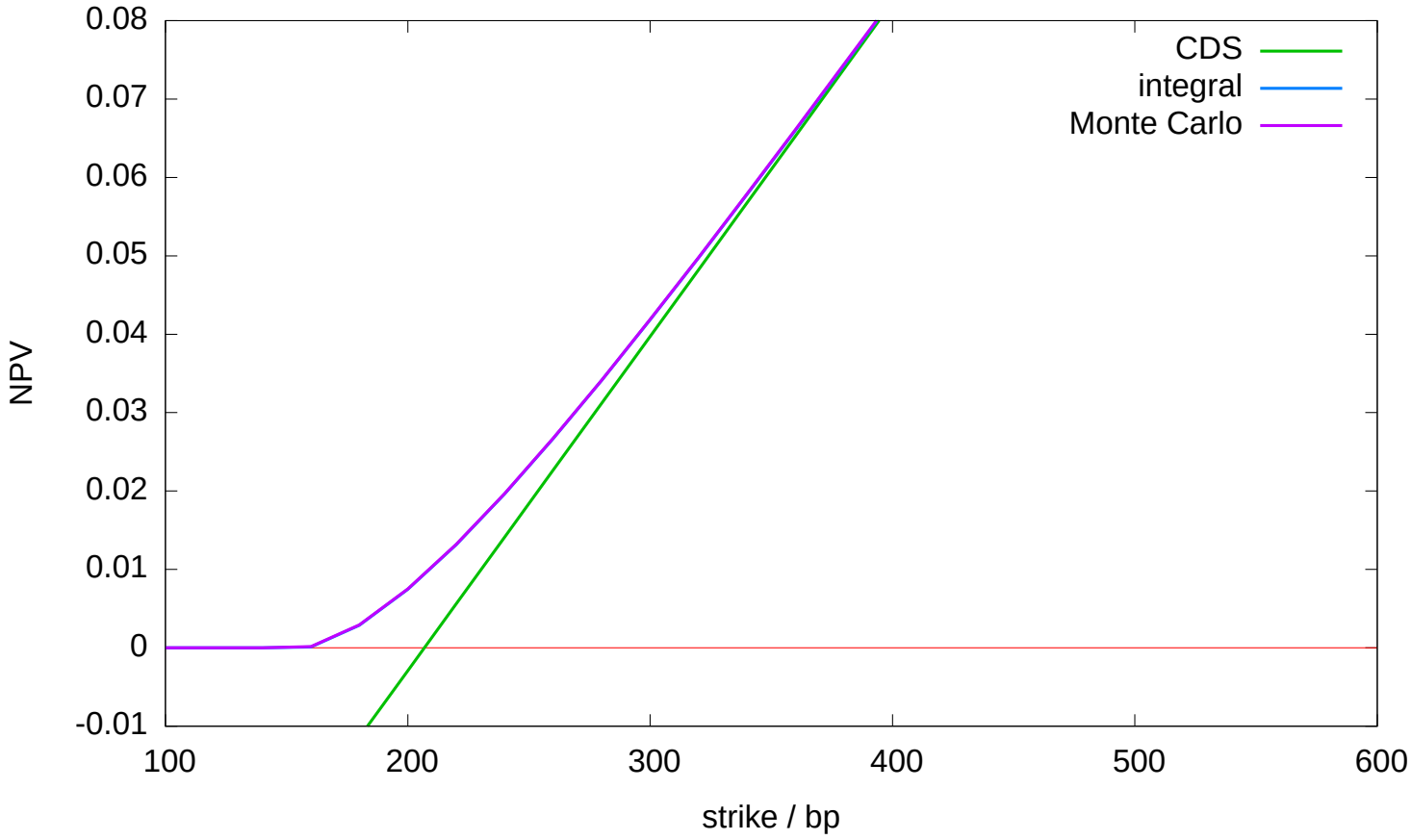
For CDS we use in general the original QuantLib **MidPointCdsEngine**. However we have also created bespoke version of a CDS pricing engine to deal with the (model implied) default curve in the background.

New CDS-Option pricing engines are added:

- CIR model - Numerical Integral
- CIR model - Monte Carlo
- CIR model - Analytic
- Jump CIR model - Monte Carlo
- Jump CIR model - Semi Analytic

CIR CDS Option Pricing Engine

CDS and CDS Option prices as a function of strike: Protection **seller**



Calibration Instruments

In general we exploit original QuantLib instruments for CDS and CDSO.

For the calibration, CdsCalibrationHelper and CdsoCalibrationHelper were added, both derived from **CalibrationHelper**.

In case of the shifted version (JCIR++), we don't need to use credit default swaps, as we are matching the termstructure by construction. We can concentrate on credit default swap options only.

In case of the unshifted version (JCIR), we need to include credit default swaps as well, to fit to the market termstructure.

Calibration Optimisation

During the calibration we minimize the absolute price error, in QuantLib words:

CalibrationHelper::CalibrationErrorType errorType = CalibrationHelper::PriceError

Making usage of the "calibrate" method of the **CalibratedModel** base class.

We use our adaptive simulated annealing technique as optimisation method. Hereby QuantLib components such as EndCriteria are used. The non-negativity constraint will be considered, as well as the parameter specification, of which parameter to be affected.

Calibration Example

Credit index using the JCIR++ model

- Underlying CDS 5Y at ∼90bp
- Market (Black) volatilities 40-44% for strike 90bp (∼ATM)

Calibrated parameters:

$a = 0.195873,\ \theta = 0.012001,\ \sigma = 0.068567,\ y_0 = 0.013487,\ 2a\theta/\sigma^2 \approx 1$
$\alpha = 0.004584,\ \gamma = 0.449476$

Implied vs Market vols (ATM):

Expiry	Model NPV	Market NPV	Implied Vol	Market Vol
1M	15.89	16.94	38.10	40.63
2M	25.31	27.42	40.40	43.77
3M	33.91	34.88	42.21	43.42
4M	39.46	39.33	43.27	43.13
5M	45.28	44.47	44.29	43.50

Scenario generation

We choose - in view of multi-name simulations - a numerical scheme for y (λ) propagation following Alfonsi (2005) and recommendation in Brigo (2006):

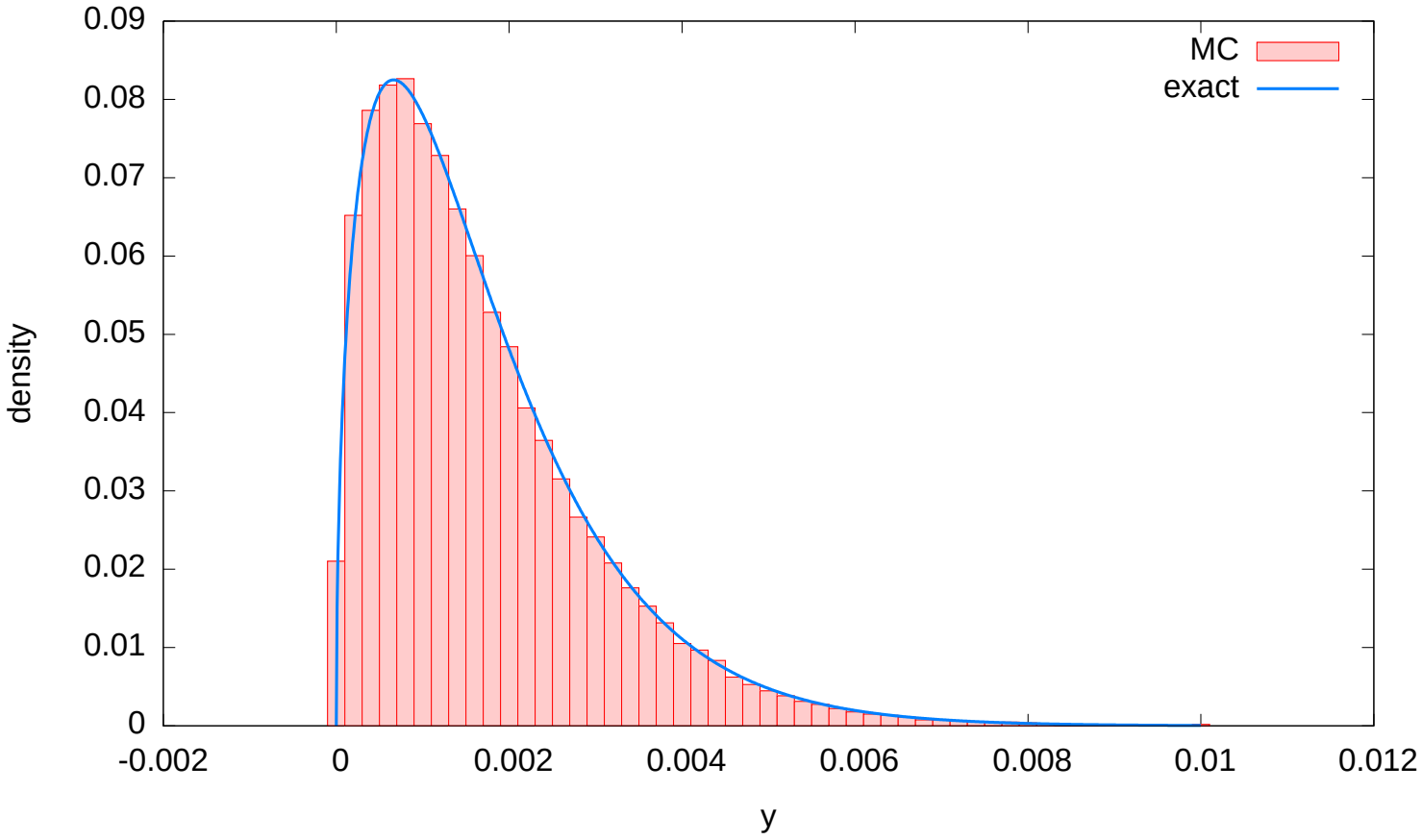
$$y_{i+1} = \left(\left(1 - \frac{a}{2}(t_{i+1} - t_i)\right) \sqrt{y_i} + \frac{\sigma \left(W_{i+1} - W_i\right)}{2\left(1 - \frac{a}{2}(t_{i+1} - t_i)\right)} \right)^2 + (a\,\theta - \sigma^2/4)(t_{i+1} - t_i)$$

Including jumps leads to the following modifications:

- We let all jumps occur at period end.
- The jump component is using the QuantLib class **InverseCumulativePoisson**, which returns the number of jumps.
- For each jump, determine jump size s using the inverse cumulative distribution function of the exponential distribution: $s = -\gamma \ln(1 - u)$, where u is a random number uniform in $(0, 1)$.

CIR: Propagation Graph

Hazard rate distribution at time $t = 10$, MC with monthly time steps



Calibrated parameters from Brigo 2006, p. 795: $a = 0.354201$, $\theta = 0.00121853$, $\sigma = 0.0238186$, $y_0 = 0.0181$ so that $2\,a\,\theta/\sigma^2 = 1.52154$.

Backtesting

We focus on the change in CDS spreads over certain horizons for certain tenors.

General idea:

- We can calibrate the model and generate (model implied) distributions (horizon/tenor)
- For each observation we identify historical changes in spreads (horizon/tenor)

We perform a goodness of fit test, whether the given (historical) sample of data is drawn from a given (model implied) probability distribution.

Candidates are:

- Anderson Darling statistic
- Exception counting
- Cramer von Mises statistic