Stochastic Local Volatility in QuantLib

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QuantLib User Meeting 2014
Düsseldorf
2014-12-06

- Heston Stochastic Local Volatility
- Fokker-Planck Equations
  - Square Root Process
  - Boundary Conditions
  - Coordinate and Density Transformations
- Calibration

Local Volatility [Dupire 1994]

- Local Volatility $\sigma_{LV}(S,t)$ as function of spot level S_t and time t:

$$
\begin{aligned}
d \ln S_t &= \left(r_t - q_t - \frac{1}{2}\sigma_{LV}^2(S,t) \right) dt + \sigma_{LV}(S,t) dW_t \\
\sigma_{LV}^2(S,t) &= \left. \frac{\frac{\partial C}{\partial T} + (r_t - q_t) K \frac{\partial C}{\partial K} + q_t C}{\frac{K^2}{2} \frac{\partial^2 C}{\partial K^2}} \right|_{K=S, T=t}
\end{aligned}
$$

- Consistent with option market prices.
- Model is often criticized for its unrealistic volatility dynamics.
- Dupire formula is mathematically appealing but also unstable.

Stochastic Volatility [Heston 1993]

- Stochastic volatility given by a square-root process:

$$\begin{aligned} d\ln S_t &= \left(r_t - q_t - \frac{1}{2}\nu_t\right)dt + \sqrt{\nu_t}\,dW_t^S \\ d\nu_t &= \kappa(\theta - \nu_t)\,dt + \sigma\sqrt{\nu_t}\,dW_t^\nu \\ \rho dt &= dW_t^\nu\, dW_t^S \end{aligned}$$

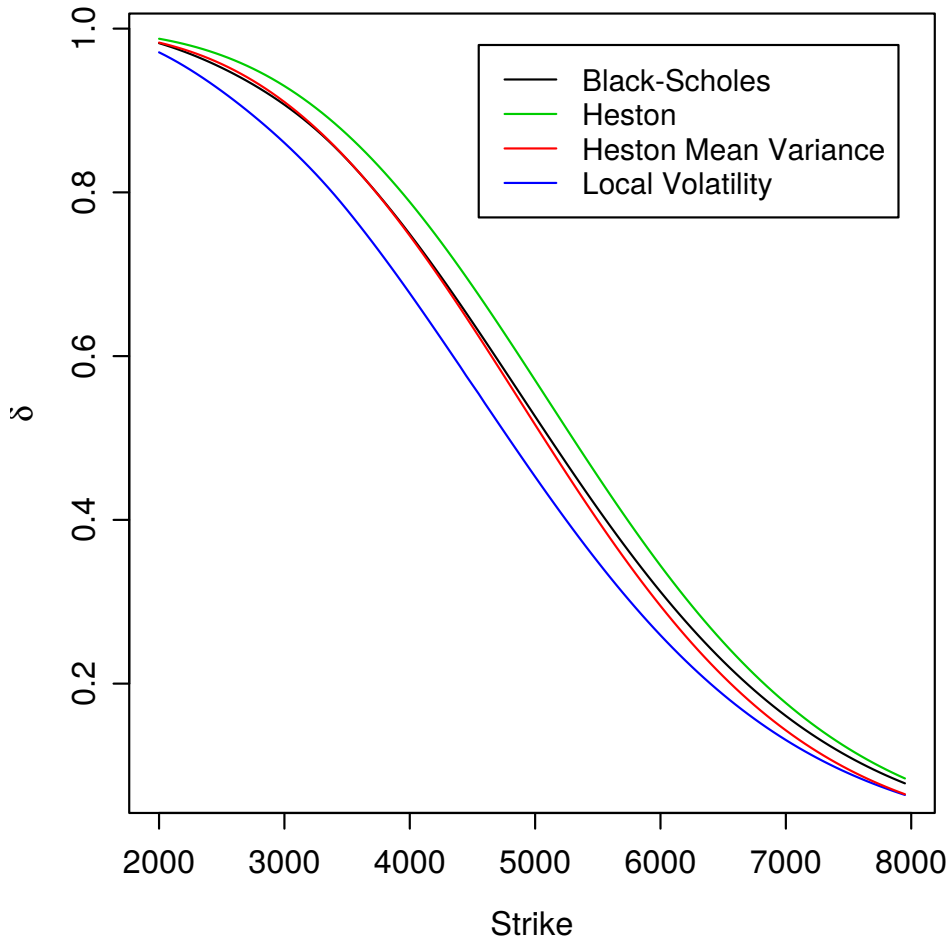
- Semi-analytical solution for European call option prices:

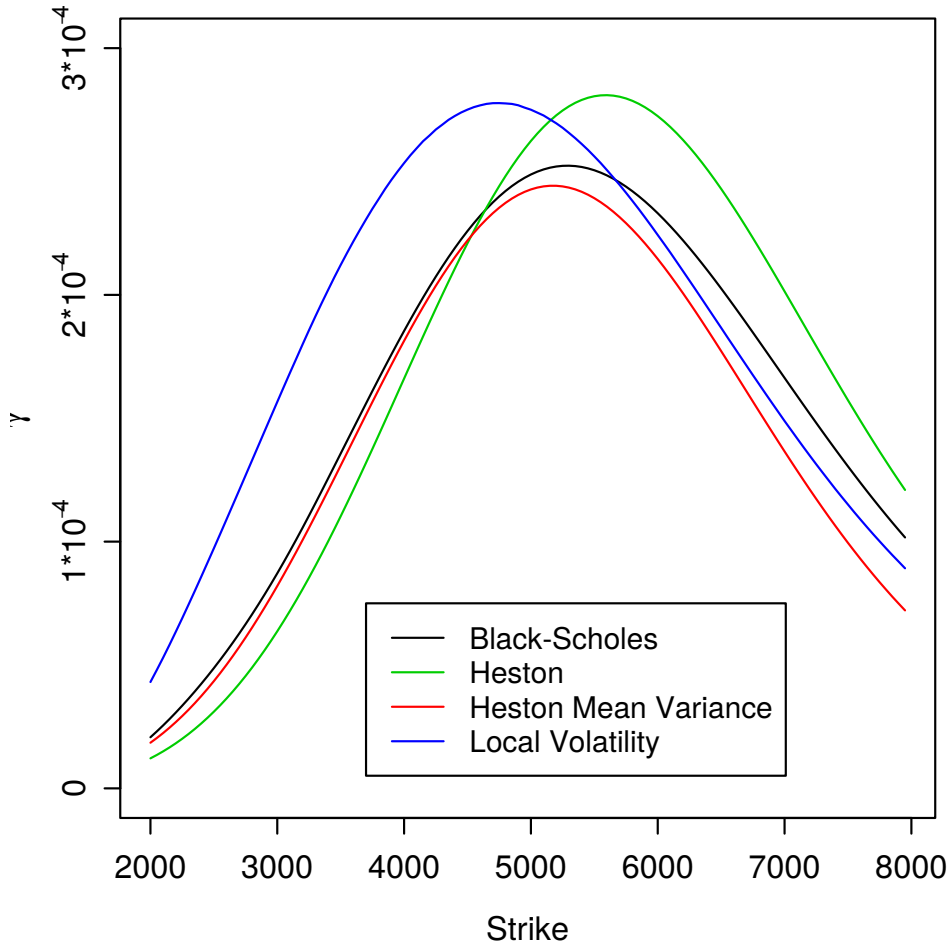
$$\begin{aligned} C(S_0, K, \nu_0, T) &= SP_1 - Ke^{-(r_t - q_t)T}P_2 \\ P_j &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{e^{-iu\ln K}\phi_j(S_0, K, \nu_0, T, u)}{iu}\right]du \end{aligned}$$

- More realistic volatility dynamics.
- Does often not exhibit enough skew for short dated expiries.

Example: Differences in δ and γ

The implied and local volatility surface is derived from the Heston model and therefore the option prices between all models match.
$S_0 = 5000, \kappa = 5.66, \theta = 0.075, \sigma = 1.16, \rho = -0.51, \nu_0 = 0.19, T = 1.7$





Heston Stochastic Local Volatility

- Add leverage function $L(S_t, t)$ and mixing factor η:

$$\begin{aligned} d \ln S_t &= \left(r_t - q_t - \frac{1}{2} L(S_t, t)^2 \nu_t \right) dt + L(S_t, t) \sqrt{\nu_t} dW_t^S \\ d\nu_t &= \kappa (\theta - \nu_t) dt + \eta \sigma \sqrt{\nu_t} dW_t^\nu \\ \rho dt &= dW_t^\nu dW_t^S \end{aligned}$$

- Leverage $L(x_t, t)$ is given by probability density $p(S_t, \nu, t)$ and

$$L(S_t, t) = \frac{\sigma_{LV}(S_t, t)}{\sqrt{\mathbb{E}[\nu_t | S = S_t]}} = \sigma_{LV}(S_t, t) \sqrt{\frac{\int_{\mathbb{R}^+} p(S_t, \nu, t) d\nu}{\int_{\mathbb{R}^+} \nu p(S_t, \nu, t) d\nu}}$$

- Mixing factor η tunes between stochastic and local volatility.

Cheat Sheet: Link between SDE and PDE

Starting point is a multidimensional SDE of the form:

$$d\boldsymbol{x}_t = \boldsymbol{\mu}(\boldsymbol{x}_t, t)dt + \boldsymbol{\sigma}(\boldsymbol{x}_t, t)d\boldsymbol{W}_t$$

Feynman-Kac: price of a derivative $u(\boldsymbol{x}_t, t)$ with boundary condition $u(\boldsymbol{x}_T, T)$ at maturity T is given by:

$$\partial_t u + \sum_{k=1}^{n} \mu_i \partial_{x_k} u + \frac{1}{2} \sum_{k,l=1}^{n} \left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)_{kl} \partial_{x_k}\partial_{x_l} u - ru = 0$$

Fokker-Planck: time evolution of the probability density function $p(\boldsymbol{x}_t, t)$ with the initial condition $p(\boldsymbol{x}, t = 0) = \delta(\boldsymbol{x} - \boldsymbol{x_0})$ is given by:

$$\partial_t p = -\sum_{k=1}^{n} \partial_{x_k} \left[\mu_i p\right] + \frac{1}{2} \sum_{k,l=1}^{n} \partial_{x_k}\partial_{x_l} \left[\left(\boldsymbol{\sigma}\boldsymbol{\sigma}^T\right)_{kl} p\right]$$

Backward Feynman-Kac Equation

The SLV model leads to following Feynman-Kac equation for a function $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}, (x, \nu, t) \mapsto u(x, \nu, t)$:

$$\begin{aligned} 0 &= \partial_t u + \frac{1}{2} L^2 \nu \partial_x^2 u + \frac{1}{2} \eta^2 \sigma^2 \nu \partial_\nu^2 u + \eta \sigma \nu \rho L \partial_x \partial_\nu u \\ &\quad + \left(r - q - \frac{1}{2} L^2 \nu \right) \partial_x u + \kappa (\theta - \nu) \partial_\nu u - ru \end{aligned}$$

- PDE can be solved using either Implict scheme (slow) or more advanced operator splitting schemes like modified Craig-Sneyd or Hundsdorfer-Verwer in conjunction with damping steps (fast).
- Implementation is mostly harmless, extend `FdmHestonOp`.

Forward Fokker-Planck Equation

The corresponding Fokker-Planck equation for the probability density $p : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, (x, \nu, t) \mapsto p(x, \nu, t)$ is:

$$\begin{aligned} \partial_t p &= \frac{1}{2}\partial_x^2 \left[L^2 \nu p\right] + \frac{1}{2}\eta^2\sigma^2\partial_\nu^2 \left[\nu p\right] + \eta\sigma\rho\partial_x\partial_\nu \left[L\nu p\right] \\ &\quad -\partial_x \left[\left(r - q - \frac{1}{2}L^2\nu\right) p\right] - \partial_\nu \left[\kappa\left(\theta - \nu\right) p\right] \end{aligned}$$

- Numerical solution of the PDE is cumbersome due to difficult boundary conditions and the Dirac delta distribution as the initial condition.
- PDE can be efficiently solved using operator splitting schemes, preferable the modified Craig-Sneyd scheme

Square Root Process

Main issues of the implementation are caused by the square root process:

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dW$$

It has the following Fokker-Planck equation for the probability density $p : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, (\nu, t) \mapsto p(\nu, t)$:

$$\partial_t p = \frac{\sigma^2}{2}\partial_\nu^2 [\nu p] - \partial_\nu [\kappa(\theta - \nu)p]$$

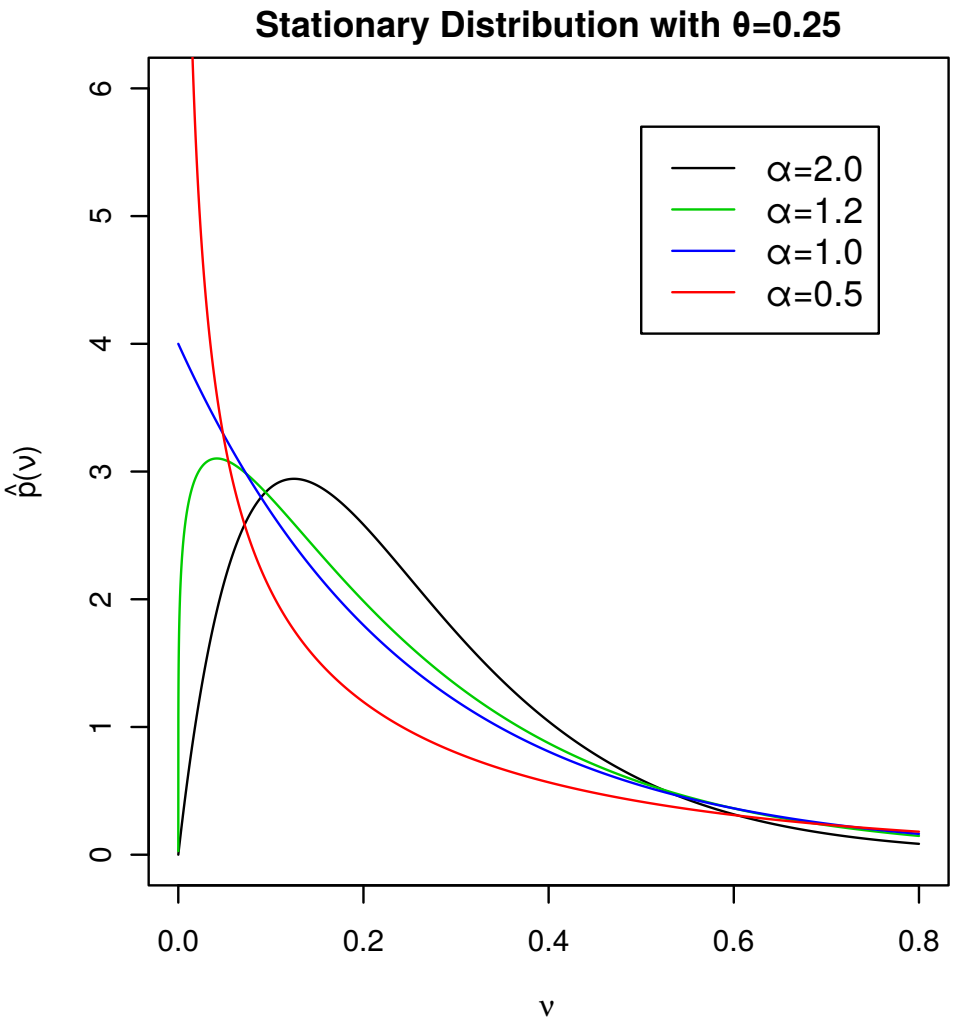
The stationary probability density $\hat{p}(\nu)$ with $\partial_t \hat{p}(\nu) = 0$ is:

$$\hat{p}(\nu) = \beta^\alpha \nu^{\alpha-1} \exp(-\beta\nu)\Gamma(\alpha)^{-1}, \alpha = \frac{2\kappa\theta}{\sigma^2}, \beta = \frac{\alpha}{\theta}$$

Stationary Probability Density

$$\lim_{\nu \to 0} \hat{p}(\nu) = \begin{cases} \infty & \text{if } \alpha < 1 \\ \theta^{-1} & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases}$$

The square root process ν_t is strictly positive if the Feller Condition $\alpha > 1$ is met.



Boundary Condition

The probability weight within $[\nu_{min}, \nu_{max}]$ of $p(\nu, t)$ is evolving by:

$$\partial_t \int_{\nu_{min}}^{\nu_{max}} d\nu p = \int_{\nu_{min}}^{\nu_{max}} d\nu \left(\frac{\sigma^2}{2} \partial_\nu^2 [\nu p] - \partial_\nu [\kappa(\theta - \nu) p] \right)$$

In order to avoid leaking of probability we enforce:

$$\partial_t \int_{\nu_{min}}^{\nu_{max}} d\nu p = 0 \quad \Rightarrow \quad \left[\frac{\sigma^2}{2} \partial_\nu [\nu p] - [\kappa(\theta - \nu) p] \right] \Bigg|_{\nu_{min}}^{\nu_{max}} = 0$$

$$\Rightarrow \quad \left[\frac{\sigma^2}{2} \partial_\nu [\nu p] - [\kappa(\theta - \nu) p] \right] \Bigg|_{\nu = \nu_{min}, \nu_{max}} = 0$$

Zero Flux Boundary Condition

Discretization

On a non-uniform grid $\{z_1, \ldots, z_n\}$ the two-sided approximation of $\partial_z f$ is:

$$\begin{aligned}\partial_z f(z_i) &\approx \frac{h_{i-i}^2 f_{i+1} + (h_i^2 - h_{i-1}^2) f_i - h_i^2 f_{i-1}}{h_{i-1} h_i (h_{i-1} + h_i)} \\ &= \frac{h_{i-1}}{h_{i-1} + h_i} \frac{f_{i+1} - f_i}{h_i} + \frac{h_i}{h_{i-1} + h_i} \frac{f_i - f_{i-1}}{h_{i-1}}\end{aligned}$$

With $h_i := z_{i+1} - z_i$ and $f_i := f(z_i)$. The second order derivative is approximated by:

$$\partial_z^2 f(z_i) \approx \frac{h_{i-i} f_{i+1} - (h_{i-1} + h_i) f_i + h_i f_{i-1}}{\frac{1}{2} h_{i-1} h_i (h_{i-1} + h_i)}$$

Discretization

Sort by factors of f_i, set

$$\begin{aligned}
\zeta_i &:= h_i h_{i-1} \\
\zeta_i^p &:= h_i(h_{i-1} + h_i) \\
\zeta_i^m &:= h_{i-1}(h_{i-1} + h_i)
\end{aligned}$$

then:

$$\begin{aligned}
\partial_z f(z_i) &\approx \frac{h_{i-i}}{\zeta_i^p} f_{i+1} + \frac{(h_i - h_{i-1})}{\zeta_i} f_i - \frac{h_i}{\zeta_i^m} f_{i-1} \\
\partial_z^2 f(z_i) &\approx \frac{2}{\zeta_i^p} f_{i+1} - \frac{2}{\zeta_i} f_i + \frac{2}{\zeta_i^m} f_{i-1}
\end{aligned}$$

Discretization

A general partial differential equation of the form

$$\partial_t f = A(z)\partial_z^2 f + B(z)\partial_z f + C(z)f$$

has therefore the spacial discretization:

$$\begin{aligned}
\partial_t f(z_i) &= \frac{2A_i + B_i h_{i-i}}{\zeta_i^p} f_{i+1} + \left(\frac{-2A_i + B_i(h_i - h_{i-1})}{\zeta_i} + C_i \right) f_i \\
&\quad + \frac{2A_i - B_i h_i}{\zeta_i^m} f_{i-1} \\
&=: \gamma_i f_{i+1} + \beta_i f_i + \alpha_i f_{i-1}
\end{aligned}$$

Discretization

This is interpreted as a tridiagonal transfer matrix T with diagonal β_i, upper diagonal γ_i, and lower diagonal α_i:

$$T := \begin{pmatrix} \beta_1 & \gamma_1 & 0 & \dots & & \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \dots & \\ 0 & \alpha_3 & \beta_3 & \gamma_3 & 0 & \dots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ & \vdots & & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} \\ & & & & \alpha_n & \beta_n \end{pmatrix}$$

Then

$$\partial_t \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = T \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Boundary Condition

Add z_0 below the lower boundary and z_{n+1} above the upper boundary to the grid. The zero flux condition takes the general form

$$[\partial_z A(z,t) f + B(z,t) f] \Big|_{z=z_0, z_{n+1}} \stackrel{!}{=} 0$$

Lower Boundary: The partial derivative is discretized by a second order forward differentiation, so that all terms are given by grid points

$$\begin{aligned} \partial_z f(z_0) &\approx \frac{-h_0^2 f_2 + (h_1+h_0)^2 f_1 - ((h_1+h_0)^2 - h_0^2) f_0}{h_0 h_1 (h_1+h_0)} \\ &= -\frac{h_0}{\zeta_1^p} f_2 + \frac{(h_0+h_1)}{\zeta_1} f_1 - \frac{(2h_0+h_1)}{\zeta_1^m} f_0 \end{aligned}$$

Boundary Condition

The general zero-flux boundary condition is therefore discretized at the lower boundary as

$$\begin{aligned} 0 &= -\frac{h_0}{\zeta_1^p} A_0 f_2 + \frac{(h_0+h_1)}{\zeta_1} A_0 f_1 + \left(-\frac{(2h_0+h_1)}{\zeta_1^m} A_0 + B_0 \right) f_0 \\ &=: c_1 f_2 + b_1 f_1 + a_1 f_0 \\ \Rightarrow f_0 &= -\frac{c_1}{a_1} f_2 - \frac{b_1}{a_1} f_1 \end{aligned}$$

$$\begin{aligned} \partial_t f_1 &= \gamma_1 f_2 + \beta_1 f_1 + \alpha_1 f_0 \\ &= (\gamma_1 - \alpha_1 \frac{c_1}{a_1}) f_2 + (\beta_1 - \alpha_1 \frac{b_1}{a_1}) f_1 \end{aligned}$$

$\rightarrow$ modification of the transfer matrix.

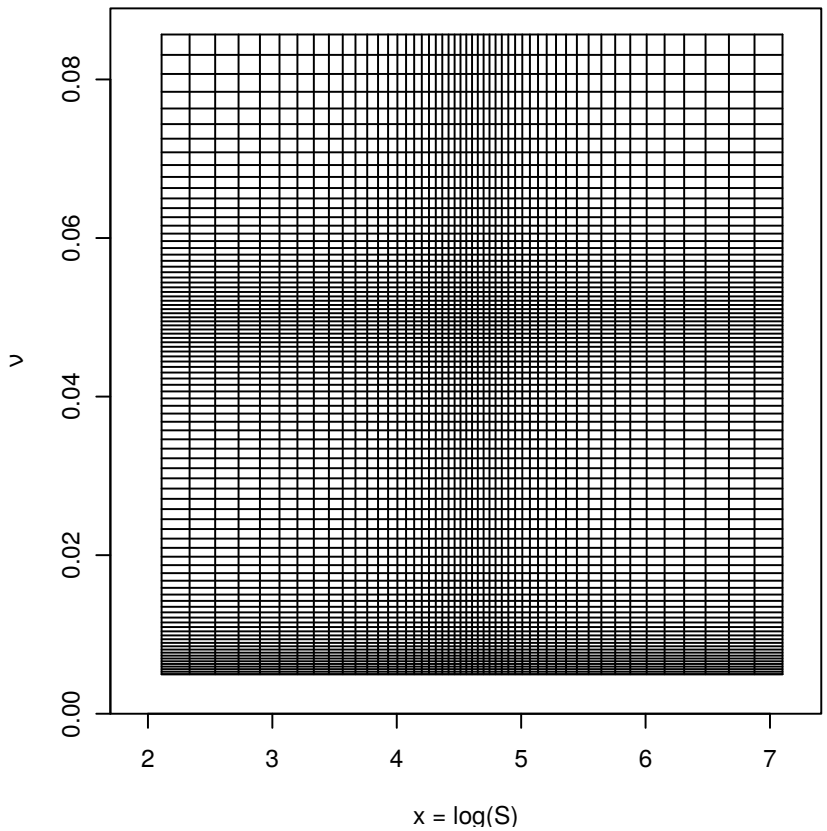
Non-Uniform Meshes

Non-uniform meshes are a key component [Tavella & Randall 2000]

Define coordinate transformation $Y = Y(\epsilon)$ for n critical points B_k with density factors β_k

$$\begin{aligned}
\frac{dY(\epsilon)}{d\epsilon} &= A\left[\sum_{k=1}^{n} J_k(\epsilon)^{-2}\right]^{-\frac{1}{2}} \\
J_k(\epsilon) &= \sqrt{\beta^2 + (Y(\epsilon) - B_k)^2} \\
Y(0) &= Y_{min} \\
Y(1) &= Y_{max}
\end{aligned}$$

ODE solver is based on Peter's Runge-Kutta implementation.



Example: $x_0 = \ln(100), \nu_0 = 0.05$, Feller constraint is fulfilled

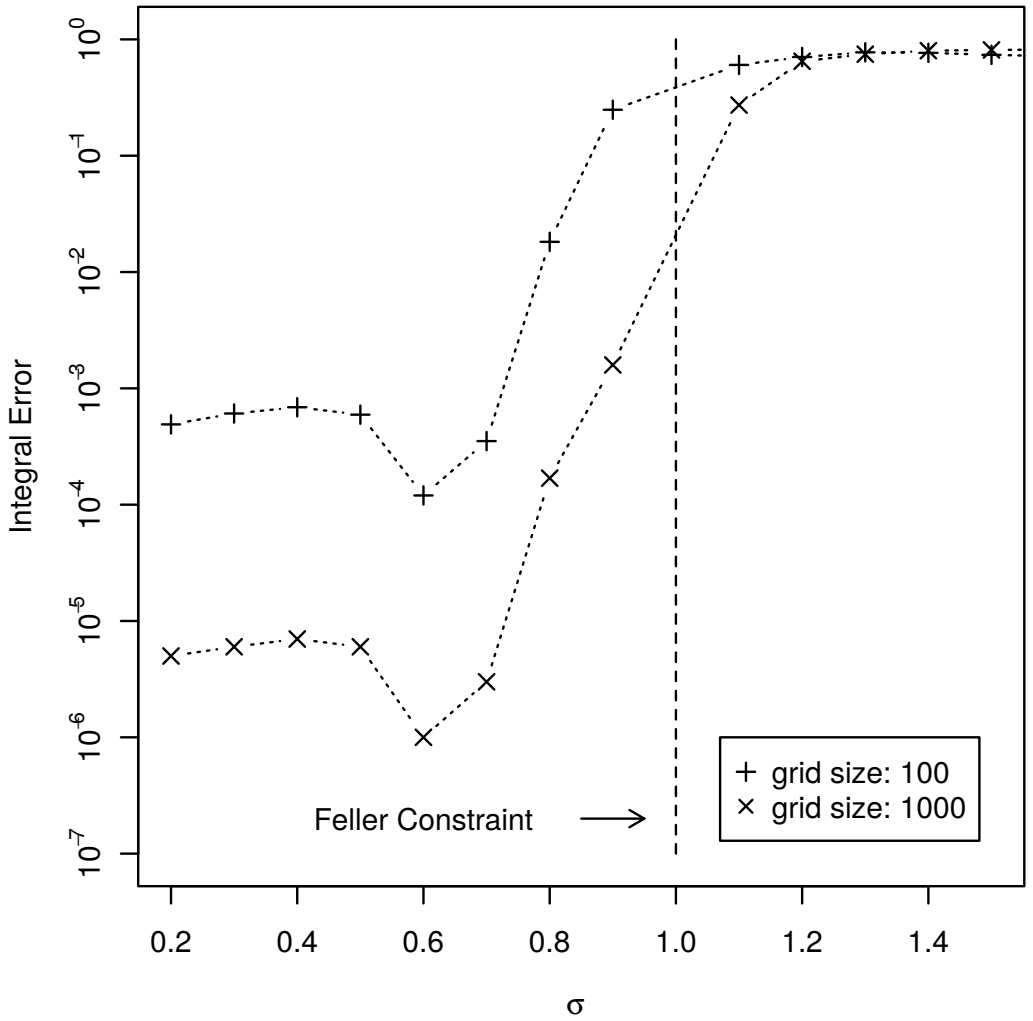
Loss of Probability

Time evolution of the stationary distribution with zero flux condition.

$$
\begin{aligned}
P(x) &= \int_{-\infty}^{x} \hat{p}(\nu) d\nu \\
\nu_{min} &= P^{-1}(0.01) \\
\nu_{max} &= P^{-1}(0.99) \\
\int_{\nu_{min}}^{\nu_{max}} \hat{p} d\nu &= 0.98
\end{aligned}
$$

Integral error after evolving for one year:

$$
\left| \int_{\nu_{min}}^{\nu_{max}} p(\nu, t = 1y) d\nu - 0.98 \right|
$$



Transformed Density

Recap: Stationary distribution:

$$\hat{p}(\nu) = \beta^{\alpha}\nu^{\alpha-1}\exp(-\beta\nu)\Gamma(\alpha)^{-1}$$

Remove divergence following Lucic [2] by using

$$\begin{aligned} q &= \nu^{1-\alpha}p \\ \Rightarrow \partial_t q &= \frac{\sigma^2}{2}\nu\partial_\nu^2 q + \kappa(\nu+\theta)\partial_\nu q + \frac{2\kappa^2\theta}{\sigma^2}q \end{aligned}$$

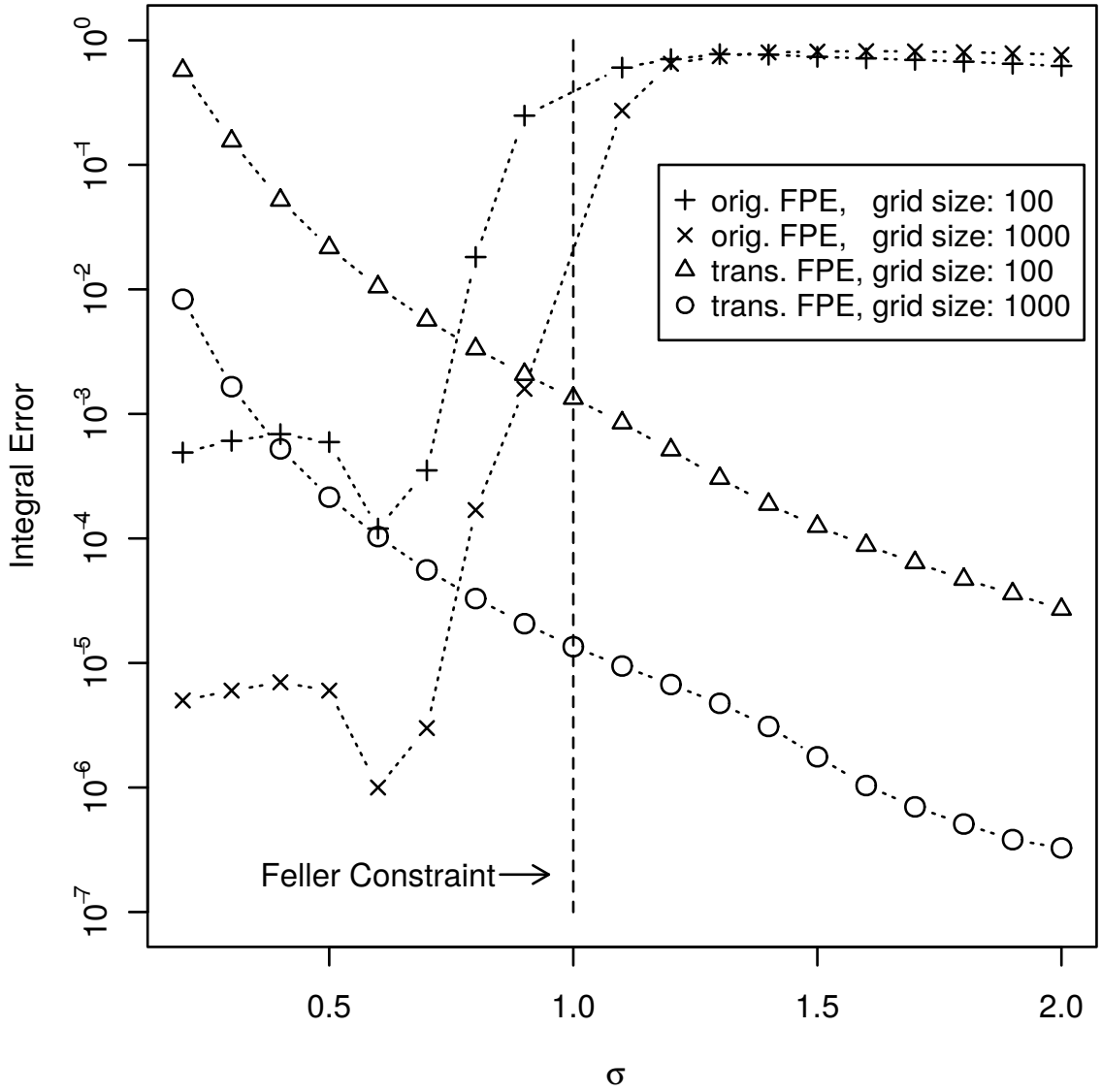
This equation has the stationary solution

$$\hat{q}(\nu) = \beta^{\alpha}\exp(-\beta\nu)\Gamma(\alpha)^{-1}$$

which converges to $\beta^{\alpha}\Gamma(\alpha)^{-1}$ as $\nu \to 0$

Transformed Probability Density

Time evolution of the transformed distribution with zero flux condition.



Log Coordinates

Apply Itô's lemma to $z = \log \nu$:

$$dz = ((\kappa\theta - \frac{\sigma^2}{2})\frac{1}{\nu} - \kappa)dt + \sigma\frac{1}{\sqrt{\nu}}dW$$

Fokker-Planck equation for the probability distribution $f : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, (z, t) \mapsto f(z, t)$ ($\nu = \exp(z)$):

$$\partial_t f(z,t) = -\partial_z((\kappa\theta - \frac{\sigma^2}{2})\frac{1}{\nu} - \kappa)f + \partial_z^2(\frac{\sigma^2}{2}\frac{1}{\nu}f)$$

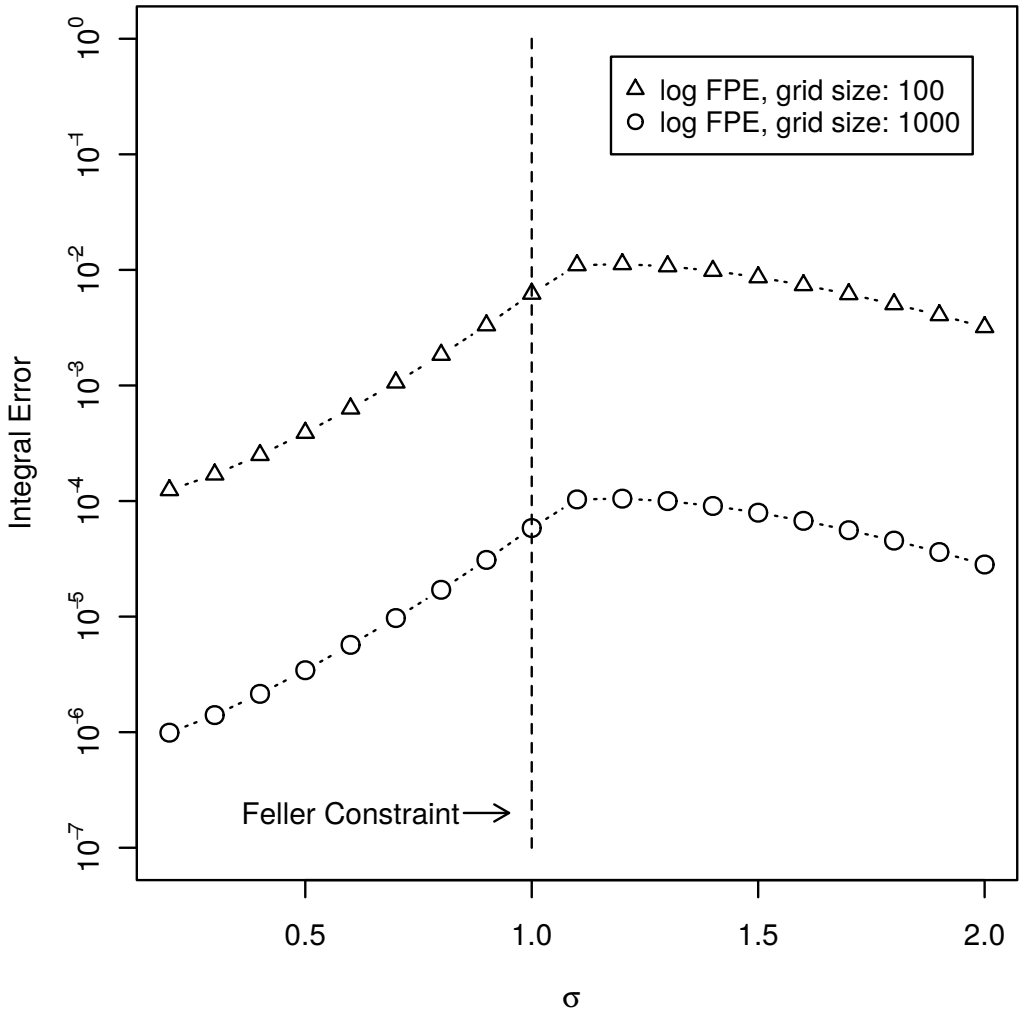
Stationary solution:

$$\hat{f}(z) = \beta^\alpha \exp(z\alpha)\exp(-\beta\exp(z))\Gamma(\alpha)^{-1} = \nu\hat{p}(\nu)$$

$\hat{f}$ converges to 0 as $z \to -\infty$

Log Coordinates

Time evolution of log probability density with zero flux condition



$$\nu_{min} = min\left(0.001, F^{-1}(0.01)\right)$$

Intermediate Results

- Proper implementation of the zero flux boundary condition is not enough to get a stable scheme.
- Transformation of the PDE in log coordinates leads to a less poisonous problem.
- Non-Uniform meshers are a key component for success.

  $\rightarrow$ all in all, mostly harmless ☺. Time for another dimension

Zero Flux Boundary Condition in two Dimensions

Adding the stock process to the picture complicates matters a bit. Probability density has a second variable $x = \log S$, and the Fokker-Planck equation reads

$$\partial_t f = \partial_z^2 A(z,x,t) f + \partial_z B(z,x,t) f + \partial_z \partial_x \rho C(z,x,t) f + \text{powers of } \partial_x$$

Stretching the argument above a bit[1] we arrive at the boundary condition

$$\left[\partial_z A(z,x,t) f + B(z,x,t) f + \rho \partial_x C(z,x,t) f\right] \Big|_{z=z_0,z_1} \stackrel{!}{=} 0$$

[1] Can be made rigorous [2]

SLV Fokker-Planck: Natural Coordinates

$$
\begin{aligned}
dx_t &= (r_t - q_t - \frac{\nu_t}{2})dt + \sqrt{\nu_t} L(x,t) dW_t^x \\
d\nu_t &= \kappa(\theta - \nu_t)dt + \eta\sigma\sqrt{\nu_t} dW_t^\nu \\
\rho dt &= dW_t^x dW_t^\nu
\end{aligned}
$$

Fokker-Planck equation:

$$
\begin{aligned}
\partial_t p &= \frac{1}{2}\partial_x^2 \left[L^2 \nu p\right] + \frac{1}{2}\eta^2\sigma^2\partial_\nu^2 \left[\nu p\right] + \eta\sigma\rho\partial_x\partial_\nu \left[L\nu p\right] \\
&\quad -\partial_x \left[\left(r - q - \frac{1}{2}L^2\nu\right) p\right] - \partial_\nu \left[\kappa\left(\theta - \nu\right) p\right]
\end{aligned}
$$

The zero flux condition takes the form $\forall x$:

$$
\left[\frac{\sigma^2}{2}\nu\partial_\nu p + \left(\kappa(\nu - \theta) + \frac{\sigma^2}{2}\right) p + \rho\nu\sigma\partial_x L p\right]\Bigg|_{\nu=\nu_0, \nu=\nu_{n+1}} = 0
$$

SLV Fokker-Planck: Transformed Density $q = \nu^{1-\alpha} p$

Fokker-Planck equation for $q = \nu^{1-\alpha} p$

$$\begin{aligned}\partial_t q &= \frac{\nu}{2}\partial_x^2 L^2 q + (-r_t + q_t)\partial_x q + \partial_x(\frac{\nu}{2}L^2 + \rho\sigma\frac{2\kappa\theta}{\sigma^2}L)q \\ &+ \frac{\sigma^2}{2}\nu\partial_\nu^2 q + \kappa(\nu+\theta)\partial_\nu q + \frac{2\kappa^2\theta}{\sigma^2}q \\ &+ \rho\sigma\nu\partial_x\partial_\nu L q\end{aligned}$$

The zero flux condition takes the form $\forall x$:

$$\left[\frac{\sigma^2}{2}\nu\partial_\nu q + \kappa\nu q + \rho\nu\sigma\partial_x L q\right]\Big|_{\nu=\nu_0,\nu=\nu_{n+1}} = 0$$

SLV Fokker-Planck: Log Coordinates

$$
\begin{aligned}
dx_t &= (r_t - q_t - \frac{\nu_t}{2})dt + \sqrt{\nu_t} L(x,t) dW_t^x \\
dz_t &= ((\kappa\theta - \frac{\sigma^2}{2})\frac{1}{\nu} - \kappa)dt + \eta\sigma\frac{1}{\sqrt{\nu}} dW_t^\nu \\
\rho dt &= dW_t^x dW_t^\nu
\end{aligned}
$$

Fokker-Planck equation:

$$
\begin{aligned}
\partial_t f = & \frac{1}{2}\partial_x^2\left[L^2\nu f\right] + \frac{\eta^2\sigma^2}{2}\partial_z^2\left[\frac{1}{\nu}f\right] + \eta\sigma\rho\partial_x\partial_z\left[Lf\right] \\
& -\partial_x\left[\left(r - q - \frac{1}{2}L^2\nu\right)f\right] - \partial_z\left[\left((\kappa\theta - \frac{\sigma^2}{2})\frac{1}{\nu} - \kappa\right)f\right]
\end{aligned}
$$

The zero-flux boundary condition is

$$
\left[\frac{\eta^2\sigma^2}{2}\frac{1}{\nu}\partial_z f - \kappa(1 - \frac{\theta}{\nu})f + \rho\sigma\partial_x Lf\right]\Big|_{\nu=\nu_0,\nu=\nu_{n+1}} \stackrel{!}{=} 0
$$

SLV Fokker-Planck: Implementation

Example log coordinates:

$$\begin{aligned}\partial_t f &= \frac{1}{2}\partial_x^2\left[L^2\nu f\right] + \frac{\eta^2\sigma^2}{2}\partial_z^2\left[\frac{1}{\nu}f\right] + \eta\sigma\rho\partial_x\partial_z\left[Lf\right] \\ &\quad -\partial_x\left[\left(r-q-\frac{1}{2}L^2\nu\right)f\right] - \partial_z\left[\left((\kappa\theta-\frac{\sigma^2}{2})\frac{1}{\nu}-\kappa\right)f\right]\end{aligned}$$

$$\begin{aligned}\partial_t f &= \frac{\nu}{2}\partial_x^2 L^2 f + \frac{\eta^2\sigma^2}{2}\frac{1}{\nu}\partial_z^2 f + \eta\sigma\rho\partial_x\partial_z Lf \\ &\quad +(-r+q)\partial_x f + \frac{\nu}{2}\partial_x L^2 f + \left[(-\kappa\theta-\frac{\sigma^2}{2})\frac{1}{\nu}+\kappa\right]\partial_z f + \frac{\kappa\theta}{\nu}f\end{aligned}$$

Use multiplication of derivative operators with L on the right hand side, added method multR to `TripleBandBinearOp` (saves some terms).

Start Condition: Dirac Delta Distribution

To begin with the Dirac delta distribution need to be regularized.

Approximation for small Δt based on

$$L(x,t) = \frac{\sigma_{LV}(x_{t=0},0)}{\sqrt{\nu_0}} = const\ \forall t \in [0,\Delta t]$$

1. Exact solution is known for $\rho = 0$
2. One Euler Step based on the SDE leads to bivariate Gaussian distribution
3. Semi-Analytical solution for exact sampling [Brodie, Kaya 2006]

Calibration

- Start with a a calibrated Local Volatility Model $\sigma_{LV}(x_t, t)$ and calibrated Heston Model $(\nu_0, \theta, \kappa, \sigma, \rho)$
- Recap: Leverage $L(x_t, t)$ is given by

$$L(x_t, t) = \frac{\sigma_{LV}(x_t, t)}{\sqrt{\mathbb{E}[\nu_t | x = x_t]}} = \sigma_{LV}(x_t, t) \sqrt{\frac{\int_{\mathbb{R}^+} p(x_t, \nu, t) d\nu}{\int_{\mathbb{R}^+} \nu p(x_t, \nu, t) d\nu}}$$

- Start condition: $p(x, \nu, 0) = \delta(x - x_0)\delta(\nu - nu_0)$

$$\Rightarrow L(x_{t=0}, 0) = \frac{\sigma_{LV}(x_{t=0}, 0)}{\sqrt{\nu_0}}$$

Calibration

Iterative Scheme:

1. Use Fokker-Planck equation to get from

$$p(x, \nu, t) \rightarrow p(x, \nu, t + \Delta t)$$

   assuming a piecewise constant leverage function $L(x_t, t)$ in t

2. Calculate leverage function at $t + \Delta t$:

$$L(x, t + \Delta t) = \sigma_{LV}(x, t + \Delta t) \sqrt{\frac{\int_{\mathbb{R}^+} p(x, \nu, t + \Delta t) d\nu}{\int_{\mathbb{R}^+} \nu p(x, \nu, t + \Delta t) d\nu}}$$

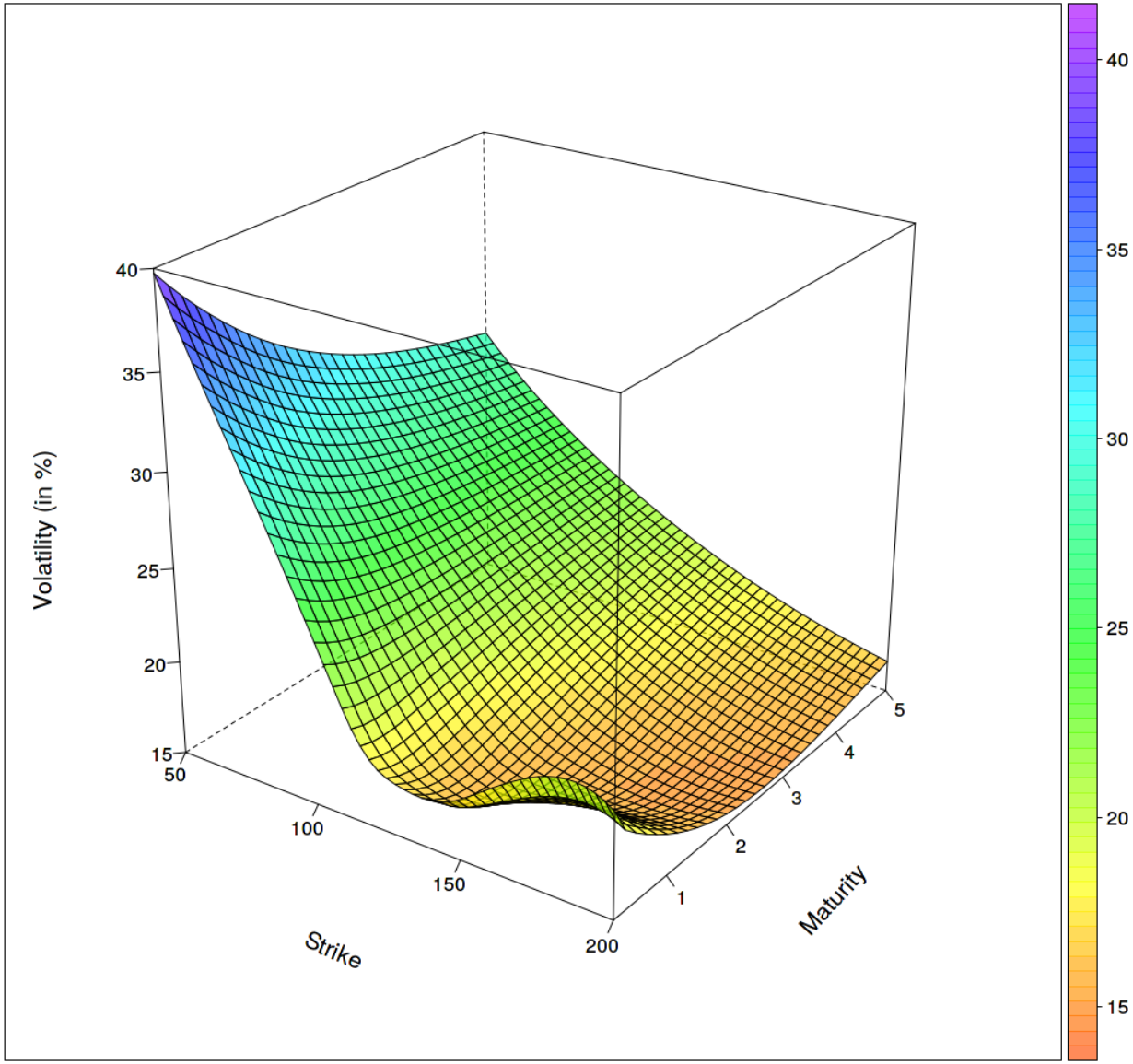
3. Set $t := t + \Delta t$
4. If t is smaller than the final maturity goto 1

Calibration Example

Motivation: Set-up extreme test case for the LSV calibration

- Feller condition is strongly violated with $\alpha = 0.6$
- Implied volatility surface of the Heston and the local volatility model differ significantly.
- Local Volatility: $\sigma_{LV}(x, t) \equiv 30\%$
- Heston Parameters:
  $S_0 = 100, \sqrt{\nu_0} = 24.5\%, \kappa = 1, \theta = \nu_0, \sigma^2 = 0.2, \rho = -75\%$
- Use log coordinates and modified Craig-Sneyd scheme

Calibration Example: Heston Implied Volatility Surface



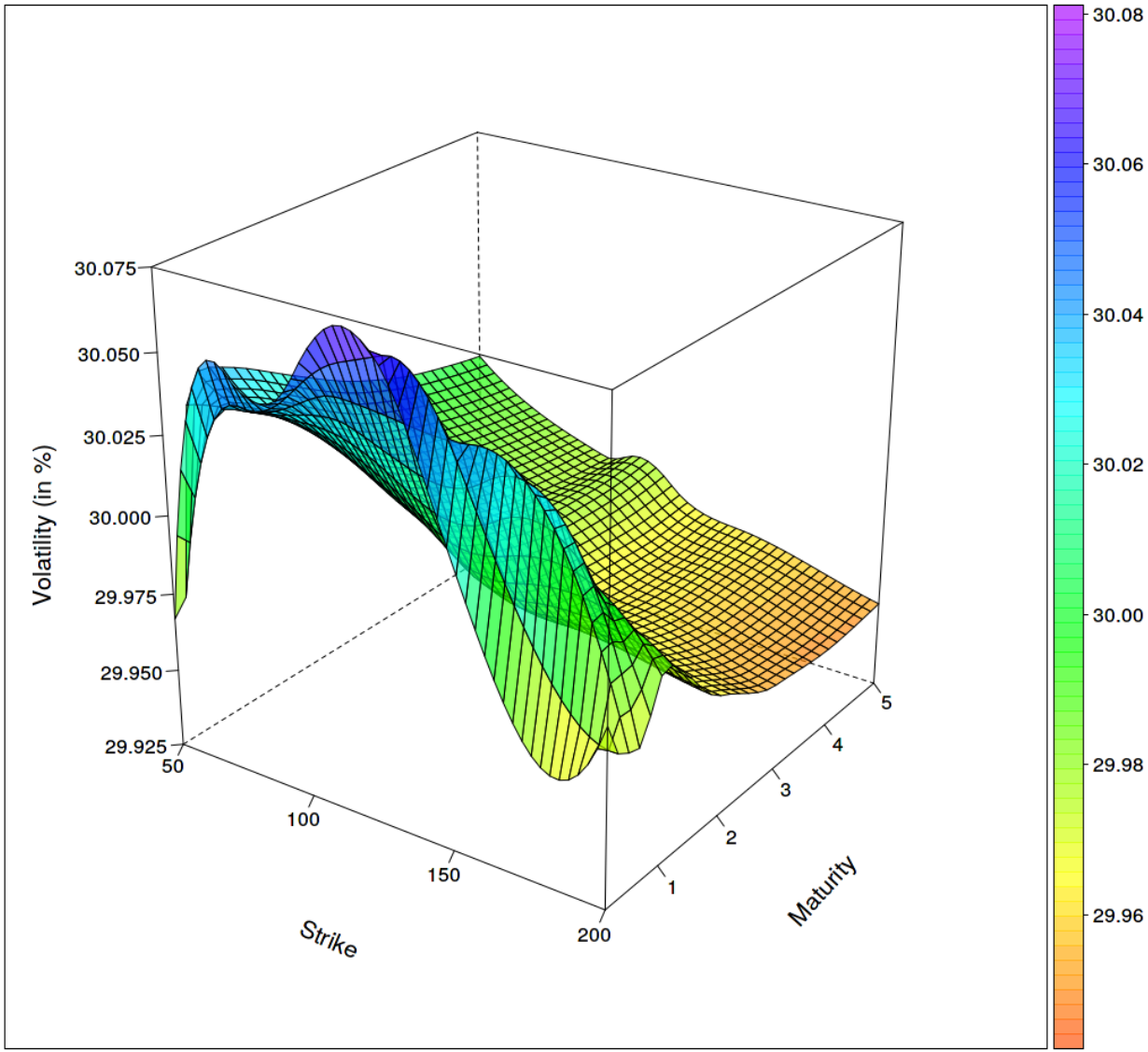
Calibration Example: Round Trip

Quality of calibration is tested by the round trip error

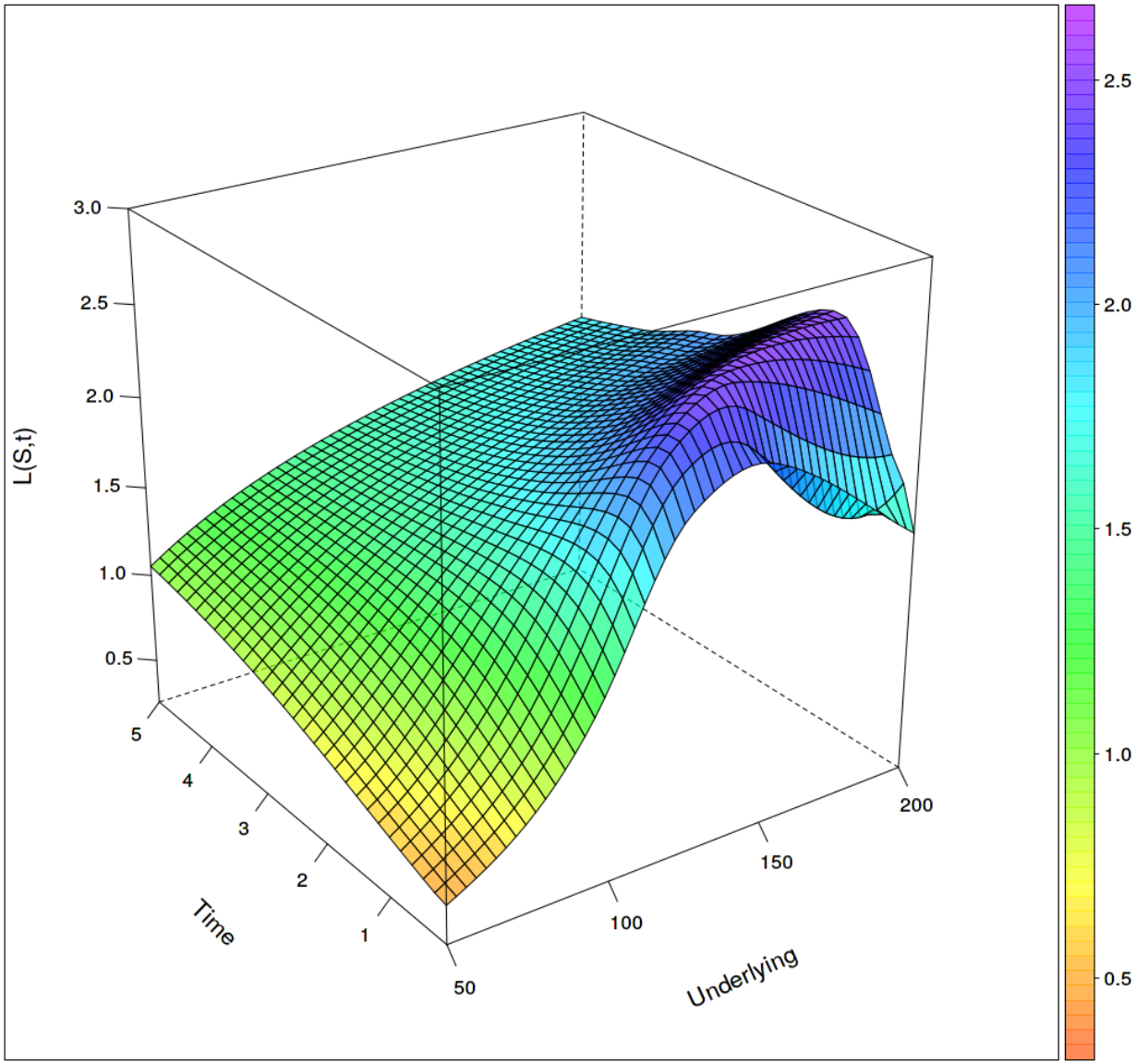
- Fokker-Planck step: Calibrate the leverage function $L(x, t)$
- Feyman-Kac step: Calculate European option prices under resulting LSV model and back out implied volatility surface
- Show differences w.r.t. expected value of

$$\sigma_{impl}(K, t) = \sigma_{LV}(S, t) = 30\%$$

Calibration Example: LSV Implied Volatility Surface



Calibration Example: Leverage Function $L(S_t, t)$



Conclusion: Heston Local Volatility in QuantLib

- ✓ Backward Feyman-Kac solver
- ✓ Forward Fokker-Planck solver
  - ✓ Zero-Flux boundary condition
  - ✓ natural and log coordinates, transformed probability density
- ✓ Non-uniform meshers are a key factor for success
- ✓ Heston Local Volatility calibration
- ✓ Round trip errors are around 5bp in vols for extreme case

Repository:

`https://github.com/jschnetm/quantlib/tree/slv/QuantLib`

Literature


William Feller.
Two singular diffusion problems.
The Annals of Mathematics, 54(1):173–182, 1951.


Vladimir Lucic.
Boundary conditions for computing densities in hybrid models via PDE methods.
Stochastics, 84(5-6):705–718, 2012.


Yu Tian, Zili Zhu, Geoffrey Lee, Fima Klebaner, and Kais Hamza.
Calibrating and Pricing with a Stochastic-Local Volatility Model., 2014.
http://ssrn.com/abstract=2182411.

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